On the Throughput, Capacity, and Stability Regions of Random Multiple Access

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***Abstract*—This paper studies finite-terminal random multiple access over the standard multipacket reception (MPR) channel. We characterize the relations among the throughput region of random multiple access, the capacity region of multiple access without code synchronization, and the stability region of ALOHA protocol. In the first part of the paper, we show that if the MPR channel is standard, the throughput region of random multiple access is coordinate convex. We then study the information capacity region of multiple access without code synchronization and feedback. Inner and outer bounds to the capacity region are derived. We show that both the inner and the outer bounds converge asymptotically to the throughput region. In the second part of the paper, we study the stability region of finite-terminal ALOHA multiple access. For a class of packet arrival distributions, we demonstrate that the stationary distribution of the queues possesses positive and strong positive correlation properties, which consequently yield an outer bound to the stability region. We also show the major challenge in obtaining the closure of the stability region is due to the lack of sensitivity analysis results with respect to the transmission probabilities. Particularly, if a conjectured "sensitivity monotonicity" property held for the stationary distribution of the queues, then equivalence between the closure of the stability region and the throughput region follows as a direct consequence, irrespective of the packet arrival distributions.**

***Index Terms*—ALOHA, capacity, multipacket reception (MPR), positive correlation, stability.**

I. Introduction

The simplicity of ALOHA random multiple access and its variations has made them the center of understanding contention in communication networks for over thirty years. Random multiple access has the main advantage of allowing a common channel to be dynamically shared by a group of terminals while maintaining a low level average transmission delay. However, random multiple access leads to unavoidable "collisions," in the sense that when packets from multiple terminals overlap, serious interference often results in a low or even zero packet reception probability. The occurrence of packet collisions increases with traffic. This consequently results in a major limit on the throughput of ALOHA systems. With an assumption of the collision channel model, interests in throughput analysis and stability issues of ALOHA systems can be dated back to its original proposal in the early 1970s [1], [2]. Although the stability problem for ALOHA systems with bufferless terminals has been extensively studied and is now well understood [1], [3]–[5], stability of slotted ALOHA systems with finite number of buffered terminals turned out to be complicated due to the interacting of multiple queues. Previous analysis in the literature resulted in the determination of the stability region for the two-terminal and three-terminal cases. For systems with more than three terminals, only inner and outer bounds to the stability region are available.

Stability analysis for the buffered ALOHA system over the collision channel was initiated by Tsybakov and Mikhailov in 1979 [6], where they obtained a sufficient condition for stability. The exact sufficient and necessary conditions for ergodicity of the two-terminal system and ergodicity of the symmetric system with all terminals having the same input rate and the same transmission probability were also derived. In 1988, Rao and Ephremides [7] explicitly introduced the technique of dominant systems. Under the assumption of fixed packet transmission probabilities, improved sufficient conditions and necessary conditions for stability were obtained. In 1994, Szpankowski [8] found the necessary and sufficient conditions for stability of ALOHA systems with more than two terminals. The stability region of the three-terminal system was obtained. However, for systems with more than three terminals, the necessary and sufficient condition cannot be computed explicitly since it involves the joint stationary statistics of the queues. In 1999, Luo and Ephremides [9] identified that the queues in ALOHA systems possess a special property, called the *stability ranks*. With the help of the special property, tight inner bounds and outer bounds of the stability region were derived. In all the above works, the purpose was to find the stability region of the packet arrival rate vectors with respect to the given packet transmission probabilities. For the two-terminal case, if one takes the closure of the stability regions over all possible transmission probabilities, an interesting observation is that the closure of the stability region is identical to the throughput region.[1] In 1991, under the assumption of a specific correlated packet arrival distribution, Anantharam derived the closure of the stability region of a general ALOHA system with more than two terminals [21]. Although the result obtained in [21] is still a special case which assumes a particular correlated packet arrival distribution, the expressions of the closure of the stability region and the throughput region again appear identical.

[1]This is defined as the capacity region in [3].

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All the above works were carried out under the assumption of the collision channel model, which is a simple but insightful model for the wired network environment. However, such a model often fails to characterize the wireless environment, where ambient noise and channel fading become more serious on one hand [10], and on the other hand, packets might be able to survive a collision due to the increased freedom in the space domain [11]. The first attempt to model the wireless channel in random multiple access was made by Ghez, Verdú, and Schwartz in 1988 [14], where they proposed a general symmetric multipacket reception (MPR) channel model [14], [15] and analyzed the stability issue of ALOHA systems under infinite-user and single-buffer assumption. The stability of finite-terminal ALOHA system over the capture channel, being a special case of the MPR channel, was analyzed by Sant and Sharma in 2000 [13]. Recently, Naware, Mergen, and Tong [16] studied the stability and delay of finite-terminal ALOHA systems over the asymmetric MPR channel. The closure of the stability region of the two-terminal system was derived explicitly using the idea of dominant systems. For the two-terminal system, it was shown that when the MPR channel is strong, the ALOHA system is stable with transmission probabilities of both terminals equal to one. In other words, even if a central scheduler is available, no scheduling is needed and the ALOHA random access is already stable and delay optimal [16]. Similar features were also found for the symmetric systems with more than two terminals if all terminals have the same arrival rate and the same transmission probability. For ALOHA systems with more than two terminals over the general MPR channel, [16] derived an inner bound of the stability region, which is indeed the throughput region of random multiple access. In addition to the results on stability and delay, [16] is the first work that considers the asymmetric MPR channel. Compared with the symmetric MPR channel model, the asymmetric MPR channel model is much more general in characterizing the asymmetric nature of wireless networks.

In parallel to the discussions on the stability of ALOHA systems, research works that look at random multiple-access systems from an information-theoretic point of view have also taken place [17]. The primary purpose of such research is to understand the fundamental impact of the burstiness in the source and the lack of code coordination on the capacity of multiple-access channels. Back in 1976, Gallager studied the source burstiness and the protocol information in [18]. It was pointed out that certain information rate can be achieved by encoding the idle and nonidle status of the transmitter only. Additional examples and capacity results corresponding to encoding the timing of the packets were presented in [19]. It was shown in [19] that, due to the use of timing information, the information capacity of a random-access system is higher than its service rate, which is indeed the throughput of the system. Besides the study of timing channels, in 1985, Massey and Mathys [20] derived the zero-error capacity region of random multiple access over the collision channel under the assumptions of no code synchronization among terminals and no feedback from the receiver. An interesting observation is that, the zero-error capacity region derived in [20], in number of packets per slot, has the same expression as the throughput region of random multiple access over the collision channel, which also equals the closure of the stability region in those special cases where the stability region is known [7], [8], [21]. In the meantime, Hui studied the information capacity region of the collision channel in random multiple access [22]. The result on the information capacity region showed a close relationship to the zero-error capacity region derived in [20]. Consequently, the reason why these regions have the same or approximately the same closed-form expressions becomes of a particular interest [17]. On the other hand, there have been several works in extending the above results to the wireless environment. Recently, Tinguely *et al.* found the zero-error capacity region of the recovery channel, which is indeed a special MPR channel without feedback [23]. The results were related to the information capacity region in [24]. In [25], Médard *et al.* studied the information capacity region of a multiple-access system with slotted and packetized transmissions over the additive white Gaussian noise (AWGN) channel. However, there has been no prior work that considers the information capacity region of random multiple access over the general MPR channel under the assumption of no feedback and no code synchronization, as [22], [20] did for the collision channel and [23], [24] did for the recovery channel.

In this paper, we address the relations among the throughput, capacity, and stability regions in radom multiple access, all over a "standard" and possibly asymmetric MPR channel (see a detailed definition in Section II). First, we derive the throughput region of random multiple access and show that if the MPR channel is standard, the throughput region is coordinate convex. Second, we derive both inner and outer bounds of the information capacity region of random multiple access without code synchronization and feedback. We show that both bounds approach the throughput region asymptotically as the packet size approaches infinity. Consequently, we define the asymptotic capacity region, and show that the asymptotic capacity region equals the throughput region, in number of packets per slot. Such a result explains, under a general channel model, the relation between the results of [22], [20] and the throughput region (see definition in Section III).

Next, we study the stability region of finite-terminal slotted ALOHA multiple access over the standard MPR channel. We follow the framework presented by Anantharam in [21] and show that, if the ALOHA system is stable, for a class of packet arrival distributions, the stationary distribution of the queue status possesses positive and strong positive correlation properties. Our results extend the work of [21] in the following two aspects. First, we assume the general standard MPR channel. Second, our results no longer depend on *correlated* packet arrivals, which has been considered "unrealistic" in [17],[16], [26]. Nevertheless, unlike the collision channel case, for slotted-ALOHA system over the standard MPR channel, the strong positive correlation property only gives an outer bound to the stability region. The major challenge in obtaining the closure of the stability region is due to the lack of sensitivity analysis results with respect to the transmission probabilities. To justify this comment, we present an additional property of the stationary distribution of the queues, together with a conjectured "sensitivity monotonicity" property. Under the

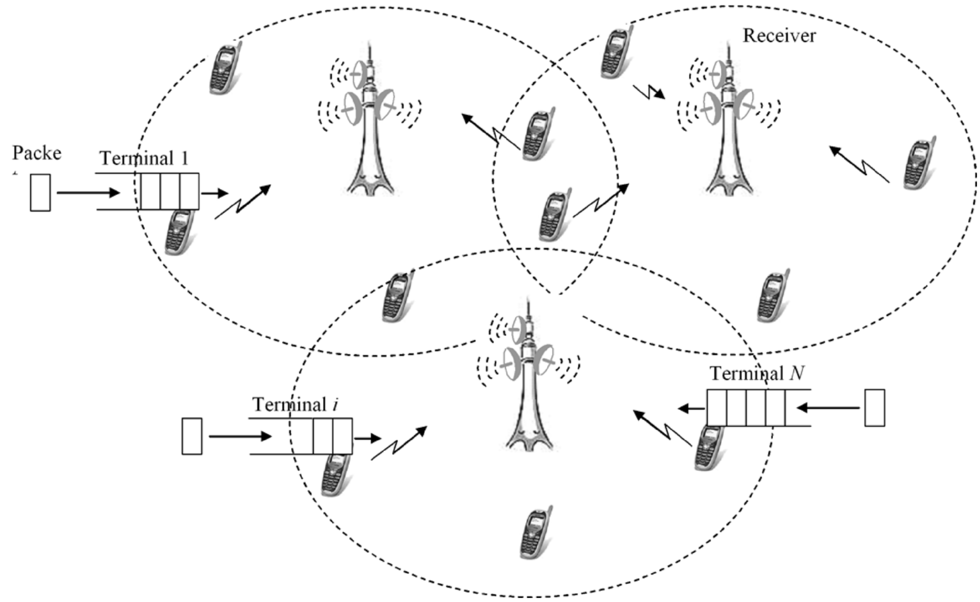

Fig. 1. Random multiple access in the control channel of a cellular network.

assumption of the conjectured sensitivity monotonicity, we show that the closure of the stability region equals the throughput region, irrespective of the packet arrival distributions. The analysis on the stability region also showed an interesting connection to the distributed scheduling protocol [26], which achieves high throughput with decentralized implementation. Further discussions on the stability issues are also presented. Although theoretical support to the conjectured property is not available, we hope the discussion can serve as a brief survey that introduces both the interesting and the challenging sides of the open stability problem, and can also serve as a guidance to future research efforts.

The rest of the paper is organized as follows. The slotted MPR channel model with finite number of terminals is described in Section II. The throughput region of random multiple access is given in Section III, where we show the throughput region is coordinate convex if the MPR channel is standard. The capacity region of random multiple access is analyzed in Section IV. By deriving inner and outer bounds of the capacity region and considering their asymptotic behavior, we prove that the asymptotic capacity region equals the throughput region. In Section V, we study the stability region of finite-terminal ALOHA multiple access. We first derive an outer bound to the stability region for a class of packet arrival distributions, by following the framework of [21]. Next, in Section V-E, we present a sensitivity monotonicity conjecture to the stationary distribution of the queues. Under the validity of this conjecture, we show the closure of stability region equals the throughput region, irrespective of the packet arrival distributions.

II. System Model

We focus on abstract channel models for the medium access control (MAC) layer of wireless networks. In the following subsection, we first clarify several special properties of the system we consider.

A. Preliminaries

There are two major properties of the MAC layer in a data network. First, information is transmitted in the form of packets. The channel model we consider in this paper is abstracted at the packet level in order to avoid excessive physical layer detail. Second, one of the key functions of the MAC layer is to transform the raw transmission facility into a logical error-free link to the upper layers. To achieve that, packets usually contain redundancy checks which, if passed, ensures very low probability of error. If a packet does not pass the redundancy check, however, it is usually dropped without being forwarded to the upper layer.

Random multiple access is one of the indispensable medium-access schemes in practical data networks. It is commonly used in the control channel of the cellular system and *ad hoc* networks [12]. The key assumption in random multiple access is that terminals transmit packets in opportunistic fashion without having full coordination among each other. Such assumption is justified in practical scenarios either due to the lack of global information, or due to the intolerable delay associated with coordination establishment. Because of opportunistic transmission, packet reception can experience serious interference from overlapping transmissions.

The wireless environment also poses two special features to the channel model we consider. First, due to the increased ambient noise power, packet reception is probabilistic even if packets do not overlap at the receiver. Second, if a packet overlaps with other packets at its receiver, different packets usually generate different amounts of interference. In other words, the channel can be asymmetric with respect to the terminals. Although the feature of probabilistic packet reception has been extensively studied in the literature, less attention has been paid to the asymmetry nature of the wireless channel. Nevertheless, it is one of the most common properties of a wireless channel, which can arise due to the use of mulituser detection, due to the use of directional antenna, or due to the natural effect of packet

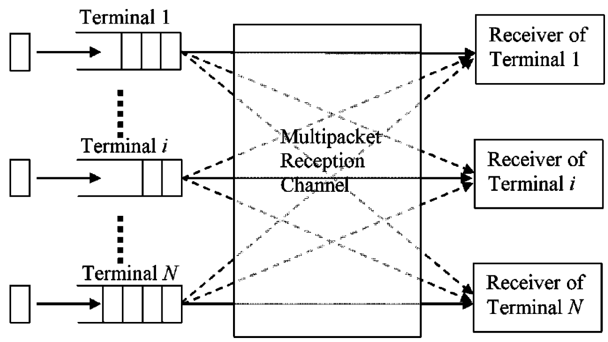

Fig. 2. Finite-terminal random multiple access over an MPR channel.

capture. A typical scenario that contains the mixture of these situations is illustrated in Fig. 1.

B. The Standard Multipacket Reception Channel

The schematic system model we study in this paper is illustrated in Fig. 2. There are N terminals. Each terminal occasionally transmits packet to its assigned receiver.[2] We assume all packets are of the same length. Time is slotted where each slot equals one packet duration, and packet transmissions start only at slot edges. In each slot, when one or more packets are transmitted, each of them has certain probability of being received successfully, or not being received by its receiver, depending on the channel model.

For a particular slot, let t_i be the transmission indicator of terminal i; $t_i = 1$ indicates that terminal i transmits a packet in the slot, while $t_i = 0$ indicates no packet transmission from terminal i in the slot. Let r_i be the reception indicator at the receiver of terminal i; $r_i = 1$ indicates a successful reception of the packet from terminal i, and $r_i = 0$ otherwise. Let $\mathcal{S}$ and $\mathcal{R}$ be two groups of terminals such that $\mathcal{R} \subseteq \mathcal{S}$. An MPR channel is specified by the complete set of parameters $\mathsf{q}_{\mathcal{R},\mathcal{S}}$, for all $\mathcal{S}$ and $\mathcal{R}$; and $\mathsf{q}_{\mathcal{R},\mathcal{S}}$ is defined by

$$\mathsf{q}_{\mathcal{R},\mathcal{S}} = P\left(r_{i\in\mathcal{R}} = 1, r_{i\notin\mathcal{R}} = 0 | t_{i\in\mathcal{S}} = 1, t_{i\notin\mathcal{S}} = 0\right). \quad (1)$$

$\mathsf{q}_{\mathcal{R},\mathcal{S}}$ can be interpreted as the probability that only and all the packets from terminals in $\mathcal{R}$ are received by their corresponding receivers given that only and all terminals in $\mathcal{S}$ transmit. We can specify a random-access channel with the complete set of the $\mathsf{q}_{\mathcal{R},\mathcal{S}}$ parameters. Such a channel model was originally introduced in [16] as the general MPR channel model, without posing any additional constraint on the packet reception probabilities.

In this paper, however, we focus on a class of MPR channels, which we define as the "standard" MPR channel. Suppose $\mathcal{U}$, $\mathcal{S}$, $\hat{\mathcal{S}}$ are three groups of terminals. We say the MPR channel is standard when the following inequality holds for all $\mathcal{U} \subseteq \mathcal{S} \subseteq \hat{\mathcal{S}}$:

$$\sum_{\mathcal{R},\, \mathcal{U}\subseteq\mathcal{R}\subseteq\mathcal{S}} \mathsf{q}_{\mathcal{R},\mathcal{S}} \geq \sum_{\mathcal{R},\, \mathcal{U}\subseteq\mathcal{R}\subseteq\hat{\mathcal{S}}} \mathsf{q}_{\mathcal{R},\hat{\mathcal{S}}}. \quad (2)$$

In other words, we assume that for the reception of any particular group of packets, simultaneous packet transmissions are not helpful.

[2]Note that multiple terminals can be assigned to the same receiver.

As defined in [15], [16], we say that the MPR channel is symmetric if $\mathsf{q}_{\mathcal{R},\mathcal{S}}$ depends only on the numbers of terminals in $\mathcal{S}$ and $\mathcal{R}$. As an example, the collision channel, defined by $\mathsf{q}_{\mathcal{S},\mathcal{S}} = 1$, if $|\mathcal{S}| = 1$, and $\mathsf{q}_{\mathcal{R},\mathcal{S}} = 0$, if $|\mathcal{S}| > 1$, is both standard and symmetric.

It can be easily seen that most of the wireless channels in random multiple access are standard. However, they are not often symmetric due to the diversity introduced in the space domain.

III. Throughput Region of Random Multiple Access

Suppose each terminal has an infinite number of packets to transmit to its receiver. In each time slot, terminal i transmits a packet with probability p_i; and with probability $1 - p_i$, terminal i keeps silent. Let $r_i \in \{0, 1\}$ be the reception indicator at the receiver of terminal i. Suppose packet transmissions are independent both among different terminals and among different time slots. Define $\boldsymbol{p}$ as the transmission probability vector, whose ith component is p_i. Given the channel parameters, the throughput of terminal i, in number of packets per slot, is defined as $T_i(\boldsymbol{p}) = P(r_i = 1)$, which is given by

$$T_i(\boldsymbol{p}) = \sum_{\substack{\mathcal{S},\mathcal{R},\\ i\in\mathcal{R}\subseteq\mathcal{S}}} \mathsf{q}_{\mathcal{R},\mathcal{S}} \prod_{j\in\mathcal{S}} p_j \prod_{k\notin\mathcal{S}} (1 - p_k). \quad (3)$$

Let $\boldsymbol{T}$ be the throughput vector, whose ith component is T_i. The "throughput region" $\mathcal{C}_T$ is defined as the union of $\boldsymbol{T}$ over all possible transmission vectors, i.e.,

$$\mathcal{C}_T = \left\{\tilde{\boldsymbol{T}} | \tilde{T}_i = T_i(\boldsymbol{p}), 0 \leq p_i \leq 1, \forall i\right\}. \quad (4)$$

The following lemma gives an important property of the throughput region of random multiple access over the standard MPR channel.

Lemma 1: If the MPR channel is standard, the throughput region $\mathcal{C}_T$ is coordinate convex, i.e., for any vector $\tilde{\boldsymbol{T}}$, if there exists a transmission vector $\boldsymbol{p}$ such that $0 \leq \tilde{T}_i \leq T_i(\boldsymbol{p})$, $\forall i$, then $\tilde{\boldsymbol{T}} \in \mathcal{C}_T$.

The proof of Lemma 1 is given in Appendix A.

IV. Capacity Region of Random Multiple Access

In this section, we study the capacity region of the standard MPR multiple-access channel from an information-theoretic point of view. In this context, we use the term "random multiple access" to refer to the assumption of no code synchronization among the terminals and no feedback from the receivers to the transmitters, as assumed in [20]. We say a rate vector $\boldsymbol{R}$ is achievable, or inside the capacity region, if an information rate R_i from terminal i to its receiver can be achieved, in the information-theoretic sense, simultaneously for all i. Unfortunately, to obtain the exact expression of the information capacity is not a trivial task. Alternatively, we first derive an outer bound and an inner bound to the capacity region in Sections IV-A and B, respectively. We later show in Section IV-C that these bounds are asymptotically equal.

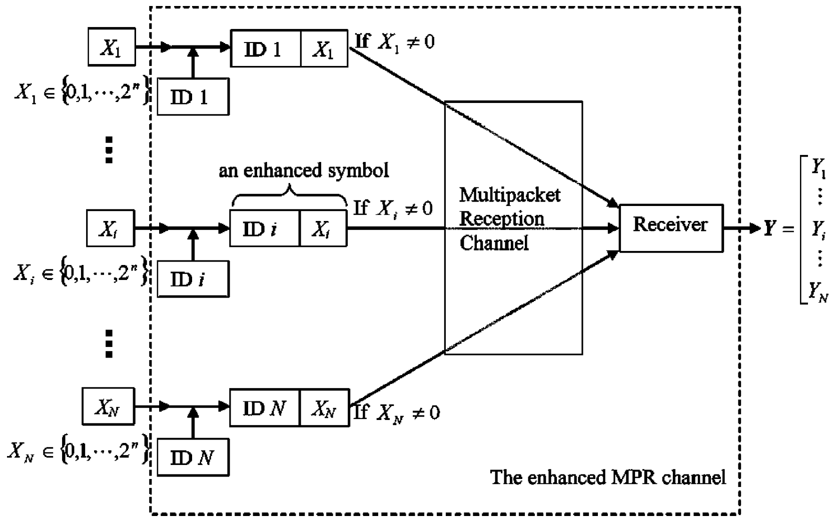

Fig. 3. The enhanced random multiple access.

A. An Outer Bound to the Capacity Region

We first construct an "enhanced" system whose capacity region contains the capacity region of the original system. The enhanced random multiple-access system is illustrated in Fig. 3.

We assume that the receivers of all terminals can exchange information instantly; or equivalently speaking, we assume all the packets are transmitted to a single receiver. We assume the code symbol alphabet for transmitter (or terminal) i is $X_i \in \{0, 1, \dots, 2^m\}$, where symbol 0 represents an idle and symbols $1 \sim 2^m$ each represent a packet of m-bit length. We further assume that when a source symbol is generated by a terminal, an "enhanced" symbol is formed automatically by attaching the transmitter ID to the data symbol. If the source symbol is $X_i = 0$, terminal i idles in the slot, while if $X_i \neq 0$, the enhanced symbol containing both the data symbol X_i and the transmitter ID is transmitted to the receiver through the MPR channel. The output of the receiver is represented by an N-length column vector $\boldsymbol{Y}$, whose ith component Y_i takes value in $\{0, 1, \dots, 2^m\}$. If a nonzero symbol X_i from terminal i is received, we have $Y_i = X_i$, otherwise, $Y_i = 0$.

Given the input symbols X_i, $1 \le i \le N$, define the set of transmitters $\mathcal{S}$ such that $X_{i \in \mathcal{S}} \neq 0$ and $X_{i \notin \mathcal{S}} = 0$. Suppose the output symbol is $\boldsymbol{Y}$. Define the set $\mathcal{R}$ such that $Y_{i \in \mathcal{R}} \neq 0$ and $Y_{i \notin \mathcal{R}} = 0$. The conditional probability of the output symbol $\boldsymbol{Y}$ is given by

$$P(\boldsymbol{Y}|X_{1 \le i \le N}) = \begin{cases} \mathsf{q}_{\mathcal{R},\mathcal{S}}, & \mathcal{R} \subseteq \mathcal{S}, Y_{i \in \mathcal{R}} = X_i, Y_{i \notin \mathcal{R}} = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

We assume the MPR channel, defined by the parameter set $\mathsf{q}_{\mathcal{R},\mathcal{S}}$, is standard.

As shown in [22], due to the lack of code synchronization and treating signals from other terminals as memoryless noise, reliable communication for each terminal i at rate $\mathcal{R}_i$, in m bits per slot, is achievable if and only if for some input distribution $P_{X_j}(X_j)$, $\forall j$, and

$$P(\boldsymbol{Y}, X_{1 \le i \le N}) = P(\boldsymbol{Y}|X_{1 \le i \le N}) \prod_{j=1}^{N} P_{X_j}(X_j) \tag{6}$$

we have, for all i

$$R_i \le \frac{1}{m} I(X_i; \boldsymbol{Y}). \tag{7}$$

Let $\tilde{X}_i \in \{1, \dots, 2^m\}$ be the nonzero symbols of terminal i. Given the source distribution $P(X_i)$, let the distribution of $\tilde{X}_i$ be $P(\tilde{X}_i) = P(X_i | X_i \neq 0)$. Let $t_i \in \{0, 1\}$ be the indicator of $X_i \neq 0$, the source symbol X_i can be written as the product of two independent random variables

$$X_i = t_i \tilde{X}_i. \tag{8}$$

Let $r_i \in \{0, 1\}$ be the indicator that $Y_i \neq 0$. The following lemma gives the information capacity region of the enhanced system.

Lemma 2: For the enhanced random multiple-access system, given the source distributions $P(X_i)$, let $\boldsymbol{p}$ be the probability vector whose ith component is $p_i = P(X_i \neq 0)$. The mutual information between X_i and $\boldsymbol{Y}$ is given by

$$I(X_i; \boldsymbol{Y}) = I(t_i; \boldsymbol{r}) + H(\tilde{X}_i) T_i(\boldsymbol{p}) \tag{9}$$

where $I(t_i; \boldsymbol{r})$ is the mutual information between t_i and $\boldsymbol{r}$; $H(\tilde{X}_i)$ is the entropy of $\tilde{X}_i$; and $T_i(\boldsymbol{p})$ is defined as in (3).

Given $\boldsymbol{p}$ and m, define $I_i^E(\boldsymbol{p}, m) = I(t_i; \boldsymbol{r}) + m T_i(\boldsymbol{p})$. The information capacity region of the enhanced system is given by

$$\mathcal{C}_I^E(m) = \left\{ \boldsymbol{R} \middle| R_i \le \frac{1}{m} I_i^E(\boldsymbol{p}, m), 0 \le p_i \le 1, 1 \le i \le N \right\}. \tag{10}$$

The proof of Lemma 2 is given in Appendix B.

B. An Inner Bound to the Capacity Region

To obtain an inner bound to the capacity region, we construct a "constrained" system, whose capacity region is contained in the capacity region of the original system. The constrained system is illustrated in Fig. 4.

We assume no information exchange between the receivers corresponding to different transmitters. Let $\lceil \log N \rceil$ be the smallest integer that is greater than or equal to $\log N$.

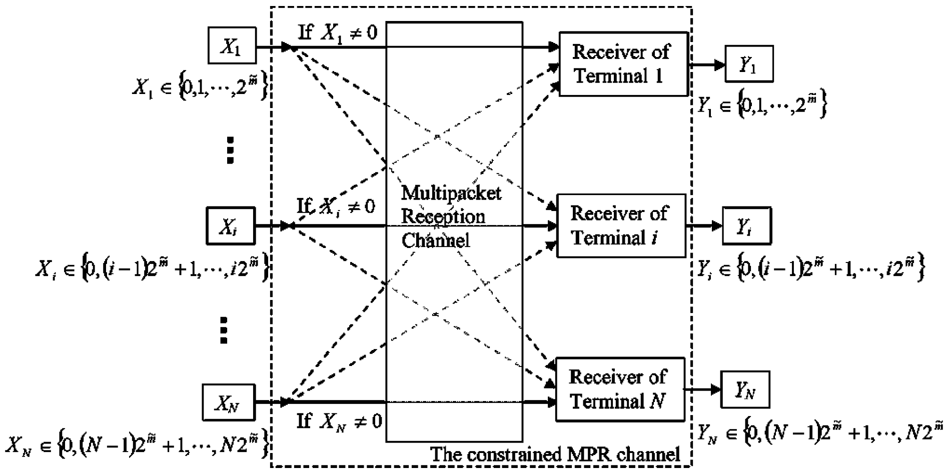

Fig. 4. The constrained random multiple access.

Assume $\tilde{m} = m - \lceil \log N \rceil \geq 0$. Instead of considering $X_i \in \{0, 1, \dots, 2^m\}$, we assume the source symbol of terminal i, X_i, is constrained within the set $\{0, (i-1)2^{\tilde{m}}+1, \dots, i2^{\tilde{m}}\}$. Let $\tilde{X}_i \in \{(i-1)2^{\tilde{m}}+1, \dots, i2^{\tilde{m}}\}$ be the nonzero symbols of terminal i, and $P(\tilde{X}_i) = P(X_i | X_i \neq 0)$. We can see that the symbol sets of $\tilde{X}_i$ and $\tilde{X}_j$, for $i \neq j$, are mutually exclusive.

At the receiver side, if a nonzero symbol $\tilde{X}_i$ within the set $\{(i-1)2^{\tilde{m}}+1, \dots, i2^{\tilde{m}}\}$ is received by the receiver of terminal i, we have $Y_i = \tilde{X}_i = X_i$; otherwise, $Y_i = 0$. Let $\boldsymbol{Y}$ be a column vector whose ith symbol is Y_i. The conditional distribution of $\boldsymbol{Y}$ given the input symbols is given by (5). Again, we assume the MPR channel, defined by the parameter set $\mathsf{q}_{\mathcal{R},\mathcal{S}}$, is standard.

Let t_i, r_i, and $\boldsymbol{p}$ be defined as in Section IV-A; the following lemma gives the capacity region of the constrained random multiple-access system.

Lemma 3: For the constrained random multiple-access system, given $\boldsymbol{p}$, $\tilde{m}$, and the source distributions $P(X_i)$, the mutual information between X_i and Y_i is given by

$$I(X_i; Y_i) = I(t_i; r_i) + H(\tilde{X}_i) T_i(\boldsymbol{p}) \tag{11}$$

where $I(t_i; r_i)$ is the mutual information between t_i and r_i; $H(\tilde{X}_i)$ is the entropy of $\tilde{X}_i$; and $T_i(\boldsymbol{p})$ is defined as in (3).

Define

$$I_i^C(\boldsymbol{p}, m) = I(t_i; \boldsymbol{r}) + (m - \lceil \log N \rceil) T_i(\boldsymbol{p}).$$

The information capacity region of the constrained system is given by

$$\mathcal{C}_I^C(m) = \left\{ \boldsymbol{R} \middle| R_i \leq \frac{1}{m} I_i^C(\boldsymbol{p}, m), 0 \leq p_i \leq 1, 1 \leq i \leq N \right\}. \tag{12}$$

The proof of Lemma 3 is presented in Appendix C.

C. Asymptotic Capacity Region of Random Multiple Access

For random multiple access over the standard MPR channel, we say a set of "asymptotic rates" R_i^∞, from terminal i to its receiver $\forall i$, is achievable if and only if we can find a sequence of rate sets $\{R_i^{(m)}\}$ such that, given the packet size m, the rate $R_i^{(m)}$ (in m bits per slot) is achievable from terminal i to its receiver, simultaneously for all i; and $\lim_{m\to\infty} R_i^{(m)} = R_i^\infty$, $\forall i$.[3] Define the "asymptotic capacity" region as the closure of the union of all achievable asymptotic rate vectors. The asymptotic capacity of a random multiple-access system can be derived from the fact that both the outer bound given in Lemma 2 and the inner bound given in Lemma 3 converge asymptotically to the throughput region.

Theorem 1: For random multiple access over the standard MPR channel, the asymptotic capacity region is given by

$$\mathcal{C}_C = \left\{ \tilde{\boldsymbol{R}} | R_i \leq T_i(\boldsymbol{p}), 0 \leq p_i \leq 1, \forall i \right\} \tag{13}$$

which equals the throughput region $\mathcal{C}_T$ according to Lemma 1.

Proof: The result follows from the fact that the outer bound $\mathcal{C}_I^E(m)$ and the inner bound $\mathcal{C}_I^C(m)$ are asymptotically equal, i.e.,

$$\lim_{m\to\infty} \frac{I_i^E(\boldsymbol{p}, m)}{m} = \lim_{m\to\infty} \frac{I_i^C(\boldsymbol{p}, m)}{m} = T_i(\boldsymbol{p}) \tag{14}$$

where $I_i^E(\boldsymbol{p}, m)$ and $I_i^C(\boldsymbol{p}, m)$ are defined in Lemmas 2 and 3, respectively. □

Since the information rate $R_i^{(m)}$ is measured in m bits per slot and m is the size of a packet, we say that the asymptotic rate $R_i^\infty = \lim_{m\to\infty} R_i^{(m)}$ is measured in "number of packets per slot," which is consistent with the measurement of the throughput.

D. Discussions on the Capacity Region

Due to the requirement of transforming the raw transmission facility into a logical error-free link to the upper layers, a packet is either not received by the receiver, or it is received with negligible probability of error. As a consequence of such error-controlled packet reception, a terminal occasionally obtains a perfect channel, for the duration of a packet length, from itself to its receiver. The probability that a terminal gets a perfect channel equals its throughput in random multiple access.

[3]In practical systems, increasing the packet size may consequently result in a change in the MPR channel parameters. Therefore, the asymptotic capacity region obtained in Theorem 1 should be interpreted as an approximation to the actual capacity region in the situation of large m.

The information rate in a random multiple-access system contains the mixture of three basic components. The first component is the information rate that can be achieved by encoding the idle and nonidle status only [18], [19]. We call this portion of the information rate the "burstiness rate" as the information is delivered by exploiting the timing of packet transmissions. This contradicts the common misunderstanding that information can only be carried by packets' data bits. The second component is the information rate that was used to deliver the transmitter ID information. Such an information rate requirement is usually not negligible in random multiple access due to the burstiness of the source and due to the lack of code synchronization among the transmitters. We term this portion of the information rate the "identification rate" as opposed to another common misconception that packets may be fully used to carry input data. The third part is the rest of the information carried by encoding the nonzero symbols in the packets. We call this portion of the information rate the "packet data rate." Although these three components are usually mixed together, which makes the exact analysis of information capacity a challenging task, it is not difficult to see that neither the "burstiness rate" nor the "identification rate" scales when the packet size increases. Consequently, in the situation of large packet size, the asymptotic behavior of the information capacity region is determined only by the "packet data rate."

In Sections IV-A and B, by assuming extra transmitter identification information, or implementing a simple transmitter identification strategy, the "burstiness rate" and the "packet data rate" can be easily decoupled in the expression of the mutual information, as shown in (9) and (11). Consequently, the "packet data rate" can be written in the multiplicative form of $H(\tilde{X}_i)T_i(\boldsymbol{p})$, which is the information carried by the nonzero packet symbols multiplied by the probability that the terminal obtains a perfect channel. The equivalence between the asymptotic capacity region and the throughput region is a direct consequence of the multiplicative form of the "packet data rate"; and the multiplicative form is indeed due to the packetized transmission model and the error-controlled packet reception, both are key features of the MAC layer in computer networks. In addition, since the asymptotic capacity region is *not* affected by the "burstiness rate," whether the receivers have the capability to detect "collision" does not change the result of Theorem 1. Furthermore, the connections among the packetized transmission, error-controlled reception and the asymptotic capacity of the system can be extended beyond the context of random multiple access. As an example, similar insight can be seen from the asymptotic convergences of the capacity bounds in a recent work on the capacity estimation of the nonsynchronous covert channel [27].

V. Stability Region of ALOHA Multiple Access

In this section, we study the stability region of a slotted finite-terminal ALOHA system over the standard MPR channel. The ALOHA multiple access and system stability are defined in Section V-A. In Section V-B, we extend the framework presented in [21] to the standard MPR channel. We show that for a class of packet arrival distributions, with independent geometric arrival being a typical example, the stationary queue status possesses the positive correlation property. A strong positive correlation property is derived in Section V-C, and consequently, an outer bound to the stability region is obtained. Unlike the collision channel case, for a general standard MPR channel, the strong positive correlation property does not lead to the closure of the stability region. The challenge is due to the lack of sensitivity analysis result with respect to the transmission probabilities. To justify this comment, in Section V-D, we present a conjecture that the stationary distribution satisfies a "sensitivity monotonicity" property. Under the assumption of the conjecture, we show the closure of the stability region equals the throughput region, irrespective of the packet arrival distributions.

A. ALOHA Multiple Access

We assume packet arrivals at the N terminals are stationary, with the average packet arrival rate at terminal i being λ_i packets per slot. We assume each terminal has a buffer of infinite capacity to store the incoming packets. The buffer of each terminal forms a queue of packets, where the arrival packets of each terminal are appended at the end of the corresponding queue, waiting for transmission. At the beginning of each slot, if terminal i's buffer is not empty, with a probability of p_i, terminal i transmits the first packet in the buffer; and with probability $1 - p_i$, terminal i keeps silent. The decision whether a terminal transmits its packet (given its queue being nonempty) is made independently from other terminals. When packets are transmitted, each of them can be either received successfully, or not received, with a probability depending on the MPR channel model, as described in Section II. We assume the information of a successful transmission is fed back to the source terminal instantly. If a transmission is successful, the corresponding packet is removed from the queue; otherwise, it stays in the queue. Define $q_i(n)$ as the number of packets in the queue of terminal i at the beginning of time slot n. As defined in [8], [9], given fixed packet arrival rates and transmission probabilities, queue i of the system is *stable* if

$$\lim_{n\to\infty} P\{q_i(n)\le x\} = F(x) \quad \text{and} \quad \lim_{x\to\infty} F(x) = 1. \tag{15}$$

If

$$\lim_{x\to\infty}\lim_{n\to\infty}\inf P\{q_i(n)\le x\} = 1 \tag{16}$$

the queue is called *substable* [28], or bounded in probability [29].

Given a fixed packet arrival rate vector $\boldsymbol{\lambda} = [\lambda_1,\dots,\lambda_N]^T$, we say $\boldsymbol{\lambda}$ is *stable* if one can find a transmission probability vector $\boldsymbol{p} = [p_1,\dots,p_N]^T$ such that all the queues in the corresponding system are stable. We say $\boldsymbol{\lambda}$ is *unstable* if such a transmission probability vector cannot be found. The union of all stable $\boldsymbol{\lambda}$ vectors is defined as the stability region of the ALOHA system.

B. Positive Correlation in Stationary Queue Status of ALOHA Systems

Let $\boldsymbol{Q}$ be an arbitrary set. A *partial order* "$\le$" is defined on $\boldsymbol{Q}$ if

- $\boldsymbol{q}_i \leq \boldsymbol{q}_i$ for all $\boldsymbol{q}_i \in \boldsymbol{Q}$;
- $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ and $\boldsymbol{q}_j \leq \boldsymbol{q}_i$ implies $\boldsymbol{q}_i = \boldsymbol{q}_j$ for all $\boldsymbol{q}_i, \boldsymbol{q}_j \in \boldsymbol{Q}$;
- $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ and $\boldsymbol{q}_j \leq \boldsymbol{q}_k$ implies $\boldsymbol{q}_i \leq \boldsymbol{q}_k$ for all $\boldsymbol{q}_i, \boldsymbol{q}_j, \boldsymbol{q}_k \in \boldsymbol{Q}$.

With the partial order well defined, $\boldsymbol{Q}$ is called a partially ordered set [31]. For example, if the component $\boldsymbol{q}_i$ of $\boldsymbol{Q}$ is an N-length integer-valued column vector. Denote the kth element of $\boldsymbol{q}_i$ by q_{ik}. We can define a partial order, such that $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ if $q_{ik} \leq q_{jk}$, $\forall k$. This is called the "regular partial order" in this paper.

Let $\{\boldsymbol{q}(n)\}$ be a discrete-time Markov chain, whose state space $\boldsymbol{Q}$ is a partially ordered set with finite or countably infinite number of components. Assume the Markov chain is irreducible and stationary, with stationary distribution denoted by U. Let $P(\boldsymbol{q}_i|\boldsymbol{q}_j)$ be the probability that the state changes to $\boldsymbol{q}_i$ in one transition, given the previous state being $\boldsymbol{q}_j$. We say the transitions of the Markov chain is "up or down," if $P(\boldsymbol{q}_i|\boldsymbol{q}_j) = 0$ unless either $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ or $\boldsymbol{q}_j \leq \boldsymbol{q}_i$ holds true [30].

Suppose f is a real-valued function defined on the state space $\boldsymbol{Q}$. We say f is *increasing* if for all $\boldsymbol{q}_i, \boldsymbol{q}_j \in \boldsymbol{Q}$, $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ implies $f(\boldsymbol{q}_i) \leq f(\boldsymbol{q}_j)$. Similarly, we say f is *decreasing* if for all $\boldsymbol{q}_i, \boldsymbol{q}_j \in \boldsymbol{Q}$, $\boldsymbol{q}_i \leq \boldsymbol{q}_j$ implies $f(\boldsymbol{q}_i) \geq f(\boldsymbol{q}_j)$.

If U is a probability function defined on $\boldsymbol{Q}$, where $U(\boldsymbol{q})$ is the probability of $\boldsymbol{q}$, we say U has "positive correlation" if for all bounded increasing functions f_1, f_2, the following inequality holds true [30]:

$$E_U[f_1(\boldsymbol{q})f_2(\boldsymbol{q})] \geq E_U[f_1(\boldsymbol{q})]E_U[f_2(\boldsymbol{q})]. \tag{17}$$

The following lemma is a simple extension to Harris' inequality presented in [30], [31].

Lemma 4: Let $\{\boldsymbol{q}(n)\}$ be a monotonic discrete-time Markov chain, whose state space $\boldsymbol{Q}$ is a countable partially ordered set. If the Markov chain is stationary with only up or down transitions, the stationary distribution U of the Markov chain has positive correlation.

The proof of Lemma 4 is presented in Appendix D.

Although the requirement of up or down transitions appears to be strong, we show next that, with the help of Lemma 4, for a class of packet arrival distributions, the stationary distribution of the queues of an ALOHA system over the standard MPR channel, measured before the transmission moment of each slot, has positive correlation.

Define the packet arrivals as follows. Assume "virtual packets" arrive at the system in each slot according to a geometric distribution with parameter Λ. In other words, the probability of the number of virtual packets ν in each slot satisfies

$$P(\nu = 0) = \frac{1}{1+\Lambda}, \quad \ldots, \quad P(\nu = k) = \frac{\Lambda^k}{(1+\Lambda)^{k+1}}. \tag{18}$$

Upon the arrival of each virtual packet, we generate a vector of real packets $\Delta\boldsymbol{q}$ ($\Delta q_i \geq 0$), according to a joint distribution of $P_a(\Delta\boldsymbol{q})$. We then append Δq_i packets to terminal i. If k virtual packets arrive in one slot, we generate k vectors $\Delta\boldsymbol{q}_1, \ldots, \Delta\boldsymbol{q}_k$ independently, each according to $P_a(\Delta\boldsymbol{q})$; and then append $\sum_{m=1}^{k} \Delta\boldsymbol{q}_{mi}$ packets to terminal i. Suppose the resulting average packet arrival rate at terminal i is λ_i. If the MPR channel is standard and $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_N]^T$ is stable, we have the following result.

Lemma 5: Assume the regular partial order. Assume the status of the queues is measured right before the transmission moment of each slot. The stationary distribution of queues $U(\boldsymbol{q})$ of the above system has positive correlation.

The proof of Lemma 5 is presented in Appendix E.

It is easy to see that if in the joint distribution $P_a(\Delta\boldsymbol{q})$, $P_a(\Delta q_i)$, and $P_a(\Delta q_j)$ are independent, then the corresponding packet arrival distributions of terminal i and j are also independent. Comparing Lemma 5 with [21, Lemma 2], in addition to considering the standard MPR channel, we show the positive correlation property for a broader class of packet arrival distributions, with independent geometric arrivals being a typical example.

C. Strong Positive Correlation Property and Outer Bound to the Stability Region

For ALOHA system over the collision channel, in addition to the positive correlation property, the strong positive correlation is an important property of the stationary distribution of the queues that consequently yields the theoretical closure of the stability region [21]. In this section, we show a similar property also holds for ALOHA system over the standard MPR channel.

Lemma 6: Suppose the ALOHA system is stable, the MPR channel is standard, and the packet arrivals can be modeled as in Lemma 5. In each slot, we associate a binary-valued flag to each of the terminals; let $b_i \in \{0, 1\}$ be the flag associated to terminal i. The value of the flag vector $\boldsymbol{b}$ is generated according to a joint distribution B and the flag generation in any particular slot is independent of all other events. Let Φ be the empty set. Let $\mathcal{S}$ be a set of terminals and $|\boldsymbol{b}_{\mathcal{S}}|$ be the number of 1's in vector $\boldsymbol{b}_{\mathcal{S}}$. Assume for all i, $\mathcal{S}$, $i \in \mathcal{S}$, the joint distribution B satisfies the following properties:

$$\begin{aligned} P_B(|\boldsymbol{b}_{\mathcal{S}}| \geq 2) &\leq \mathsf{q}_{\Phi,\mathcal{S}} \\ P_B(b_i = 1 \text{ or } |\boldsymbol{b}_{\mathcal{S}}| \geq 2) &\leq \mathsf{q}_{\Phi,\mathcal{S}} + \mathsf{q}_{\{i\},\mathcal{S}}. \end{aligned} \tag{19}$$

Denote $\boldsymbol{v}$ as the vector whose the ith component equals $b_i t_i$. Then, for any group of terminals $\mathcal{S}$, the stationary distribution of the queues conditioned on $\boldsymbol{v}_{\mathcal{S}} = \boldsymbol{0}$, denoted by $U(\boldsymbol{q}|\boldsymbol{v}_{\mathcal{S}} = \boldsymbol{0})$, has positive correlation.

The proof of Lemma 6 is presented in Appendix F.

Since setting $\boldsymbol{b} \equiv \boldsymbol{0}$ leads us back to Lemma 5, for the standard MPR channel, the set of joint distributions of B satisfying (19) is nonempty. In the special situation when we have a collision channel, we can define B such that $P_B(\boldsymbol{b} = \boldsymbol{1}) = 1$. Consequently, Lemma 6 becomes the strong positive correlation property presented in [21].

Based on the positive correlation property presented in Lemma 5 and the strong positive correlation property presented in Lemma 6, in the following lemma, we derive an outer bound to the stability region of the ALOHA system over the standard MPR channel.

Lemma 7: Suppose the ALOHA system is stable, the packet arrivals can be modeled as in Lemma 5, and the MPR channel is standard. Let $\mathsf{q}_0 = \min_{\mathcal{S}} \mathsf{q}_{\Phi,\mathcal{S}}$. Define P_B as the set of probability vectors, where $\boldsymbol{p}_b \in P_B$ if and only if the following inequalities are satisfied:

$$\prod_{i\in\mathcal{S}}(1-p_{bi}) + \sum_{i\in\mathcal{S}} p_{b_i} \prod_{j\in\mathcal{S}, j\neq i}(1-p_{bj}) \geq 1 - \frac{\mathsf{q}_{\Phi,\mathcal{S}} - \mathsf{q}_0}{1-\mathsf{q}_0}, \qquad \forall\mathcal{S} \tag{20}$$

$$\prod_{j\in\mathcal{S}}(1-p_{bj}) + \sum_{j\in\mathcal{S}, j\neq i} p_{b_j} \prod_{k\in\mathcal{S}, k\neq j}(1-p_{bk}) \geq 1 - \left\{\frac{\mathsf{q}_{\Phi,\mathcal{S}} - \mathsf{q}_0}{1-\mathsf{q}_0} + \frac{\mathsf{q}_{\{i\},\mathcal{S}}}{1-\mathsf{q}_0}\right\}, \quad \forall i, \mathcal{S}, i \in \mathcal{S}. \tag{21}$$

Define $\mathcal{C}(\boldsymbol{p}_b)$ as the region that

$$\mathcal{C}(\boldsymbol{p}_b) = \left\{ \text{vect}\left[(1-\mathsf{q}_0)p_i \prod_{j\neq i}(1-p_{bj}p_j)\right] : \begin{array}{c} 0 \leq p_i \leq 1 \\ 1 \leq i \leq N \end{array} \right\}. \tag{22}$$

Then, the packet arrival rate vector $\boldsymbol{\lambda}$ is located inside the following region:

$$\hat{\mathcal{C}}_S = \bigcap_{\boldsymbol{p}_b \in P_B} \mathcal{C}(\boldsymbol{p}_b). \tag{23}$$

The proof of Lemma 7 is presented in Appendix G.

Consider the following two extreme situations. On one hand, if we have a collision channel, since $\mathsf{q}_0 = 0$ and $\boldsymbol{p}_b \equiv \mathbf{1} \in P_B$, the outer bound becomes the closure of the stability region as given in [21]. On the other hand, if $\mathsf{q}_{\mathcal{S},\mathcal{S}} = 1$ for all $\mathcal{S}$ and packets never cause collision to each other, we have $\mathsf{q}_0 = 0$ and the only member in P_B is $\boldsymbol{p}_b \equiv \mathbf{0}$, in this case we have $\hat{\mathcal{C}}_S = \{\boldsymbol{\lambda} | \lambda_i \leq 1, \forall i\}$, which is again the closure of the stability region. However, for a general standard MPR channel, whether the outer bound given by Lemma 7 is a tight one depends on the channel parameters.

D. Discussions on the Stability Region

From the results presented in [7], [8], [21], [16], and the results presented in Section IV-A, for all the special cases where the closure of the stability region is known, the closure of the stability region of the ALOHA system is identical to the throughput region $\mathcal{C}_T$ given by (4). Therefore, it is of particular interest to know whether such an equivalence holds for a wider class of systems. Although the equality between the closure of the stability region and the throughput region has been widely conjectured at least for the collision channel case [26], unfortunately, the theoretical analysis turns out to be very challenging.

The existing literature considered two major ideas. The first one is the use of dominant systems [7]–[9], [16], which is based on the theory of stochastic dominance. The idea is to design a new system and couple it with the original system, so that the stationary distribution of the original system is stochastically dominated by that of the new system. Although such an idea helped in obtaining tight bounds to the stability region and deriving useful properties, it also showed a great difficulty in obtaining the closure of the stability region in a system with three or more terminals [8]. The second idea is to use the positive correlation property, which was first introduced to the study of ALOHA stability problem by Anantharam in [21]. The idea is mainly based on the application and extensions of Harris' inequality presented in [30]. However, to apply Harris' inequality, one needs to construct a Markov chain with only up or down transitions; and this leads to the requirement of special packet arrival distributions, as shown in [21] and also in this paper. Although the positive correlation property might be shown using other results in the literature of percolation and interacting particle systems [31], [34], [35], even if it can be shown without requiring special packet arrival distributions, to obtain the closure of the stability region of ALOHA system over a general MPR channel is still challenging.

The major difficulty in the ALOHA stability problem is due to the lack of mathematical tools in the sensitivity analysis [32], [33], with respect to the transmission probabilities, on the stationary distribution of the interacting queues. In other words, given a stable ALOHA system with a transmission probability vector, if we slightly modify some of the transmission probabilities, our knowledge about the consequent impact on the stationary distribution is very limited. If we review those successful examples where the closure of the stability region is obtained, either through the dominant system method or with the positive correlation idea, the importance of sensitivity analysis can be seen from the fact that all the proofs in the literature contain key steps leading to results for an ALOHA system with modified transmission probability vector. For example, in deriving the closure of stability region of a two-terminal system [7], [16], by assuming one terminal transmits dummy packets, the actual transmission probability of the other terminal is obtained. Equivalently, this constructs a new system with modified transmission probabilities, and consequently yields the closure of the stability region. In considering the ALOHA system over the collision channel from the perspective of positive correlation, Anantharam derived the key lemma [21, Lemma 1] using the strong positive correlation property. The inequality in the lemma is indeed an inequality for the ALOHA system with modified transmission probabilities, as shown in the proof of [21, Lemma 1]. Due to the small number of terminals [7], [16] or the special structure of the collision channel [21], the above methods obtained the closure of stability region with only one-step modification to the transmission probability vector. However, whether such a single-step probability modification can always result in the closure of the stability region is unknown for other channel models.

E. A Sensitivity Monotonicity Conjecture

In this subsection, we present an additional property together with a "sensitivity monotonicity" conjecture on the stationary distribution of ALOHA systems. If the sensitivity monotonicity conjecture is true, we show the equivalence between the stability region and the throughput region follows as a direct consequence.

For each slot, we define a_i as the flag of "channel availability associated to terminal i." Given a transmission status vector $\boldsymbol{t}$, with $t_{j\in\mathcal{S}}=1$ and $t_{j\notin\mathcal{S}}=0$, for an arbitrary $\mathcal{S}$, the conditional probability of a_i is given by

$$P(a_i|t_{j\in\mathcal{S}}=1,t_{j\notin\mathcal{S}}=0)=\sum_{\mathcal{R},i\in\mathcal{R}\subseteq\mathcal{S}\cup\{i\}} \mathsf{q}_{\mathcal{R},\mathcal{S}\cup\{i\}}. \quad (24)$$

Equation (24) can be interpreted as, given $\boldsymbol{t}$, if terminal i transmits a packet in the slot, the probability that the packet from terminal i can be received successfully is given by

$$P(a_i|t_{j\in\mathcal{S}}=1,\ t_{j\notin\mathcal{S}}=0).$$

We should first note that, given the transmission status of terminals other than i, the channel availability a_i is independent from the current transmission status of terminal i. For the standard MPR channel, $P(a_i|\boldsymbol{q})$ is a function of $q_{j\neq i}$, but not a function of q_i. In addition, $P(a_i|\boldsymbol{q})$ is a decreasing function in $q_{j\neq i}$.

The following lemma gives another property of the stationary distribution of the queues.

Lemma 8: Suppose the ALOHA system is stable and the MPR channel is standard. Let p_i be the transmission probability and let λ_i be the average packet arrival rate of terminal i. Let U be the stationary distribution of the queues and $P_U(a_i=1)$ be the stationary probability that "channel is available to terminal i." Then the following inequality is satisfied for all i:

$$\lambda_i \le p_i P_U(a_i=1). \quad (25)$$

On the other hand, given an ALOHA system, let $P_U(a_i=1)$ denote the stationary probability that $a_i=1$.[4] If the following inequality is satisfied for all i

$$\lambda_i < p_i P_U(a_i=1) \quad (26)$$

then the ALOHA system is stable.

The proof of Lemma 8 is presented in Appendix H.

Although the results in Lemma 8 appeared as a special case of the positive correlation property, it is indeed derived using the dominant system idea and hence do not depend on packet arrival distributions.

Next, we present a conjecture on the stationary distribution of the ALOHA system. Although we did not observe any counter example in computer simulations, the theoretical proof of the conjectured result is not available.

Conjecture (Sensitivity Monotonicity): Suppose A_1 and A_2 are two finite-terminal ALOHA systems with the same number of terminals, the same packet arrival distributions, and operating over the same standard MPR channel. Let $\boldsymbol{p}^{(A1)}$ and $\boldsymbol{p}^{(A2)}$ be the transmission probability vectors of A_1 and A_2, respectively. Define $P_{A1}(a_i=1)$ as the stationary probability of $a_i=1$ in system A_1, and define $P_{A2}(a_i=1)$ as the stationary probability of $a_i=1$ in system A_2. Then $P_{A1}(a_i=1)\ge P_{A2}(a_i=1)$ if $\boldsymbol{p}^{(A1)}\le\boldsymbol{p}^{(A2)}$.

Combined with Lemma 8, in the next theorem, we show the sensitivity monotonicity conjecture implies the equivalence between the closure of the stability region and the throughput region.

Theorem 2: Suppose the ALOHA system is stable and the MPR channel is standard. Suppose the conjectured sensitivity monotonicity holds true. Then the closure of the stability region of the ALOHA system is given by

$$\mathcal{C}_S=\{\boldsymbol{\lambda}|\lambda_i=T_i(\boldsymbol{p}),0\le p_i\le 1,\ \forall i\}=\mathcal{C}_T. \quad (27)$$

The proof of Theorem 2 is presented in Appendix I.

We want to emphasize that the conclusion in Theorem 2 is based on the validity of the conjecture. In addition, it is interesting to note that the iterative transmission probability update used in the proof of Theorem 2 is indeed similar to a recent proposal of a distributed scheduling MAC protocol [26]. Therefore, whether the conjectured sensitivity monotonicity can be proven theoretically is not only important to the ALOHA stability issue, but also important to practical decentralized MAC protocol design.

VI. Conclusion

In this paper, we studied the throughput, capacity and stability regions of random multiple access over the standard MPR channel. In the first part of the paper, we showed that, if the MPR channel is standard, the throughput region is coordinate convex. We then studied the information capacity region of random multiple access over standard MPR channel, and showed that the asymptotic capacity region equals the throughput region. In the second part of the paper, we studied the stability issue of ALOHA multiple access over standard MPR channel. An outer bound to the stability region is derived for a class of packet arrival distributions. We also presented a conjectured sensitivity monotonicity property, which, if can be proven, implies the equivalence between the closure of the stability region and the throughput region.

Appendix A
Proof of Lemma 1

Since the channel is standard, according to (3), $T_i(\boldsymbol{p})$ is monotonically increasing in p_i and monotonically decreasing in p_j, for all $j\neq i$. Now, given $\tilde{\boldsymbol{T}}$, by assumption, we can find a transmission probability vector $\boldsymbol{p}^{(0)}$, such that $0\le\tilde{T}_i\le T_i(\boldsymbol{p}^{(0)})$ for all i. Assume $\tilde{T}_i>0$, $\forall i$. Define $\boldsymbol{p}^{(1)}$ by

$$p_i^{(1)}=p_i^{(0)}\frac{\tilde{T}_i}{T_i(\boldsymbol{p}^{(0)})}\le p_i^{(0)}, \qquad \forall i. \quad (28)$$

Define the probability vector $\boldsymbol{p}^{(1i)}$ as $p_i^{(1i)}=p_i^{(0)}$ and $p_j^{(1i)}=p_j^{(1)}$, $\forall j\neq i$. According to (3), we have, $\forall i$

$$\tilde{T}_i=\frac{p_i^{(1)}}{p_i^{(0)}}T_i(\boldsymbol{p}^{(0)})\le\frac{p_i^{(1)}}{p_i^{(0)}}T_i(\boldsymbol{p}^{(1i)})=T_i(\boldsymbol{p}^{(1)}). \quad (29)$$

Let $n>0$ and define $\boldsymbol{p}^{(n+1)}$ by

$$p_i^{(n+1)}=p_i^{(n)}\frac{\tilde{T}_i}{T_i(\boldsymbol{p}^{(n)})}\le p_i^{(n)}, \qquad \forall i. \quad (30)$$

We can see, for all i and n, the following inequalities are satisfied:

$$p_i^{(n+1)}\le p_i^{(n)}, \qquad \tilde{T}_i\le T_i(\boldsymbol{p}^{(n+1)}). \quad (31)$$

Suppose there exists a $\delta>0$, such that

$$T_i(\boldsymbol{p}^{(n+1)})\ge\tilde{T}_i+\delta, \qquad \forall n. \quad (32)$$

[4]Note that even if the queues are not stable, the distribution of a_i can still be stationary, and hence, $P_U(a_i=1)$ can still be computed.

Then, since $\tilde{T}_i \le 1$, the following inequality must be true for all n:

$$p_i^{(n+1)} = p_i^{(n)} \frac{\tilde{T}_i}{T_i(\boldsymbol{p}^{(n)})} \le p_i^{(n)} \frac{\tilde{T}_i}{\tilde{T}_i + \delta} \le p_i^{(n)} \frac{1}{1+\delta}. \quad (33)$$

This implies $p_i^{(n+1)} \to 0$ as $n \to \infty$, and hence, $T_i(\boldsymbol{p}^{(n)}) \to 0$ as $n \to \infty$, which contradicts the assumed inequality (32). Therefore, we have $T_i(\boldsymbol{p}^{(\infty)}) = \tilde{T}_i$ for all i, and hence $\tilde{\boldsymbol{T}} \in \mathcal{C}_T$.

The situation when $\tilde{T}_i = 0$ for some i can be easily covered with minor modifications to the proof. □

Appendix B
Proof of Lemma 2

Suppose the source symbol distribution of terminal i is $P(X_i)$. Since $X_i = t_i \tilde{X}_i$ and $P(t_i = 1) = p_i$, the entropy of the source symbol of terminal i is given by

$$\begin{aligned} H(X_i) &= -\sum_{X_i} P(X_i)\log(P(X_i)) \\ &= -P(t_i=0)\log(P(t_i=0)) \\ &\quad -\sum_{\tilde{X}_i} P(t_i \neq 0)P(\tilde{X}_i)\log(P(t_i \neq 0)P(\tilde{X}_i)) \\ &= H(t_i) + p_i H(\tilde{X}_i). \end{aligned} \quad (34)$$

The conditional entropy $H(X_i|\boldsymbol{Y})$ is given by

$$H(X_i|\boldsymbol{Y}) = -\sum_{X_i,\boldsymbol{Y}} P(X_i,\boldsymbol{Y})\log\left(\frac{P(X_i,\boldsymbol{Y})}{P(\boldsymbol{Y})}\right). \quad (35)$$

Since if $Y_i \neq 0$, we have $P(X_i \neq Y_i, \boldsymbol{Y}) = 0$ and $\frac{P(X_i=Y_i,\boldsymbol{Y})}{P(\boldsymbol{Y})} = 1$, which implies

$$P(X_i,\boldsymbol{Y})\log\left(\frac{P(X_i,\boldsymbol{Y})}{P(\boldsymbol{Y})}\right) = 0.$$

Equation (35) can be written as

$$\begin{aligned} H(X_i|\boldsymbol{Y}) &= -\sum_{\substack{X_i,\boldsymbol{Y} \\ r_i=0}} P(X_i,\boldsymbol{Y})\log\left(\frac{P(X_i,\boldsymbol{Y})}{P(\boldsymbol{Y})}\right) \\ &= -\sum_{\substack{\boldsymbol{Y} \\ t_i=r_i=0}} P(X_i,\boldsymbol{Y})\log\left(\frac{P(X_i,\boldsymbol{Y})}{P(\boldsymbol{Y})}\right) \\ &\quad -\sum_{\substack{X_i,\boldsymbol{Y} \\ t_i=1,r_i=0}} P(X_i,\boldsymbol{Y})\log\left(\frac{P(X_i,\boldsymbol{Y})}{P(\boldsymbol{Y})}\right) \\ &= -\sum_{\substack{\boldsymbol{r} \\ t_i=r_i=0}} P(t_i,\boldsymbol{r})\log\left(\frac{P(t_i,\boldsymbol{r})}{P(\boldsymbol{r})}\right) \\ &\quad -\sum_{\substack{\tilde{X}_i,\boldsymbol{r} \\ t_i=1 \\ r_i=0}} P(\tilde{X}_i)P(t_i,\boldsymbol{r})\log\left(\frac{P(\tilde{X}_i)P(t_i,\boldsymbol{r})}{P(\boldsymbol{r})}\right) \\ &= -\sum_{\substack{t_i,\boldsymbol{r} \\ r_i=0}} P(t_i,\boldsymbol{r})\log\left(\frac{P(t_i,\boldsymbol{r})}{P(\boldsymbol{r})}\right) \\ &\quad + H(\tilde{X}_i)(p_i - T_i(\boldsymbol{p})). \end{aligned} \quad (36)$$

Since when $r_i = 1$

$$P(t_i,\boldsymbol{r})\log\left(\frac{P(t_i,\boldsymbol{r})}{P(\boldsymbol{r})}\right) = 0$$

the mutual information of $I(X_i;\boldsymbol{Y})$ is obtained from (34) and (41) as

$$\begin{aligned} I(X_i;\boldsymbol{Y}) &= H(X_i) - H(X_i|\boldsymbol{Y}) \\ &= H(t_i) - H(t_i|\boldsymbol{r}) + H(\tilde{X}_i)T_i(\boldsymbol{p}) \\ &= I(t_i;\boldsymbol{r}) + H(\tilde{X}_i)T_i(\boldsymbol{p}). \end{aligned} \quad (37)$$

In (37), the term $I(t_i;\boldsymbol{r})$ denotes the information rate achieved by encoding the idle and nonidle states only, and the term $H(\tilde{X}_i)T_i(\boldsymbol{p})$ is the information rate achieved by encoding the nonzero data symbols. It is easy to see that given $\boldsymbol{p}$ and m, the mutual information is maximized when the nonzero symbols are equal probable, i.e., $H(\tilde{X}_i) = m$, which implies $I(X_i;\boldsymbol{Y}) = I_i^E(\boldsymbol{p}, m)$. Therefore, by measuring the information rate in m bits per slot, the information capacity region is given by

$$\mathcal{C}_I^E(m) = \left\{\boldsymbol{R}\,\middle|\, R_i \le \frac{1}{m} I_i^E(\boldsymbol{p}, m),\ 0 \le p_i \le 1,\ 1 \le i \le N\right\}. \quad (38)$$

□

Appendix C
Proof of Lemma 3

Similar to the proof of Lemma 2, we have

$$H(X_i) = H(t_i) + p_i H(\tilde{X}_i). \quad (39)$$

Since $\boldsymbol{p}$ is given, the conditional entropy $H(X_i|Y_i)$ can be written as

$$H(X_i|Y_i) = -\sum_{X_i,Y_i} P(X_i,Y_i)\log\left(\frac{P(X_i,Y_i)}{P(Y_i)}\right). \quad (40)$$

Since if $Y_i \neq 0$, $P(X_i,Y_i)\log\left(\frac{P(X_i,Y_i)}{P(Y_i)}\right) = 0$, $H(X_i|Y_i)$ can be written as

$$\begin{aligned} H(X_i|Y_i) &= -\sum_{\substack{X_i \\ r_i=0}} P(X_i,Y_i)\log\left(\frac{P(X_i,Y_i)}{P(Y_i)}\right) \\ &= -P(t_i=0,r_i=0)\log\left(\frac{P(t_i=0,r_i=0)}{P(r_i=0)}\right) \\ &\quad -\sum_{\substack{X_i \\ t_i=1 \\ r_i=0}} P(X_i,Y_i=0)\log\left(\frac{P(X_i,Y_i=0)}{P(Y_i=0)}\right) \\ &= -\sum_{\substack{t_i \\ r_i=0}} P(t_i,r_i=0)\log\left(\frac{P(t_i,r_i=0)}{P(r_i=0)}\right) \\ &\quad + H(\tilde{X}_i)(p_i - T_i(\boldsymbol{p})) \\ &= H(t_i|r_i) + H(\tilde{X}_i)(p_i - T_i(\boldsymbol{p})). \end{aligned} \quad (41)$$

Consequently, the mutual information of $I(X_i;Y_i)$ is given by

$$\begin{aligned} I(X_i;Y_i) &= H(X_i) - H(X_i|Y_i) \\ &= I(t_i;r_i) + H(\tilde{X}_i)T_i(\boldsymbol{p}). \end{aligned} \quad (42)$$

Since the receiver of terminal i is only interested in symbols from terminal i, given $\boldsymbol{p}$, $I(X_i;Y_i)$ can be regarded as the mutual information between the source and the output of a single-user channel. For such equivalent single-user channel, $I(t_i;r_i)$ denotes the information rate achieved by encoding the idle and nonidle states only (under the constraint of $P(t_i = 1) = p_i$), and the term $H(\tilde{X}_i)T_i(\boldsymbol{p})$ is the information rate achieved by encoding the nonzero data symbols. Since the mutual information $I(X_i; Y_i)$ is maximized by $H(\tilde{X}_i) = \tilde{m} = m - \lceil \log N \rceil$, for all i, measuring the information rate in m bits per slot, the information capacity region of the constrained random multiple-access system is obtained as

$$\mathcal{C}_I^C(m) = \left\{ \boldsymbol{R} \middle| R_i \le \frac{1}{m} I_i^C(\boldsymbol{p}, m),\ 0 \le p_i \le 1,\ 1 \le i \le N \right\}. \tag{43}$$

□

Appendix D
Proof of Lemma 4

We construct a continuous-time Markov chain, where the transition time is assumed to be Poisson with rate 1, i.e., the time interval between two successive transitions yields the exponential distribution with rate 1. For each transition, the probability that the state changes from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ is given by $P(\boldsymbol{q}_i|\boldsymbol{q}_j)$, which is the transition probability of the discrete chain. Since the discrete-time Markov chain is stable by assumption, the continuous-time Markov chain is also stable. Define the stationary distribution of the discrete-time Markov chain as U and the stationary distribution of the continuous-time Markov chain as $\tilde{U}$. Define $P(\boldsymbol{q}(t))$ as the probability of the queue status being $\boldsymbol{q}$ at time t. Define $P([t, t+\Delta t])$ as the probability that a transition happens between time duration $[t, t+\Delta t]$ and let Δt be arbitrarily small. Similar to the discussion in [5, Sec. 3.3.2], we have

$$\begin{aligned} P_U(\boldsymbol{q}) &= P_{\tilde{U}}(\boldsymbol{q}(t)|[t, t+\Delta t]) \\ &= \frac{P_{\tilde{U}}(\boldsymbol{q}(t), [t, t+\Delta t])}{P_{\tilde{U}}([t, t+\Delta t])} \\ &= \frac{P_{\tilde{U}}(\boldsymbol{q}(t)) P_{\tilde{U}}([t, t+\Delta t]|\boldsymbol{q}(t))}{P_{\tilde{U}}([t, t+\Delta t])} \\ &= P_{\tilde{U}}(\boldsymbol{q}(t)) \end{aligned} \tag{44}$$

where the last equality is due to the independence between the transition time and the queue status. Equation (44) implies that $U(\boldsymbol{q}) = \tilde{U}(\boldsymbol{q})$ for any $\boldsymbol{q}$.

Now, since the continuous-time Markov chain is monotonic and has only up or down transitions, according to Harris' inequality [30], $\tilde{U}$ has positive correlation. Consequently, U has positive correlation. □

Appendix E
Proof of Lemma 5

Denote the queue status at the beginning of a slot, before the transmission moment, by $\boldsymbol{q}_j$; and denote the queue status after the transmission moment by $\boldsymbol{q}_i$. Define $P_d(\boldsymbol{q}_i|\boldsymbol{q}_j)$ as the departure transition probability, which is the probability of queue status changing from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ after the transmission in a particular slot. Now, we construct another discrete-time Markov chain, whose transition matrix $P(\boldsymbol{q}_i|\boldsymbol{q}_j)$ is given by

$$P(\boldsymbol{q}_i|\boldsymbol{q}_j) = \frac{\Lambda}{1+\Lambda} P_a(\boldsymbol{q}_i - \boldsymbol{q}_j) + \frac{1}{1+\Lambda} P_d(\boldsymbol{q}_i|\boldsymbol{q}_j). \tag{45}$$

The construction can be explained as follows. We associate a "arrival/departure" flag f to each transition of the new Markov chain. Each transition can either be an arrival transition or a departure transition, according to the associated flag. The "arrival/departure" flags are determined independently before the corresponding transitions. With probability $\frac{\Lambda}{1+\Lambda}$, $f = 0$ and the corresponding transition is an "arrival" transition, where the queue status changes from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ according to the arrival transition probability $P_a(\boldsymbol{q}_i - \boldsymbol{q}_j)$. With probability $\frac{1}{1+\Lambda}$, $f = 1$ and the corresponding transition is a "departure" transition, where the queue status changes from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ according to the departure transition probability $P_d(\boldsymbol{q}_i|\boldsymbol{q}_j)$.

Since the MPR channel is standard, the new Markov chain is monotonic. The transition of the chain is either up or down since the arrival transition matrix has only up transitions and the departure transition matrix has only down transitions. Consequently, according to Theorem 4, the stationary distribution of the constructed Markov chain has positive correlation. Let the stationary distribution of the queues of the constructed Markov chain be U.

Now, we derive the stationary distribution of the queues of the constructed Markov chain, measured only before the departure transitions. We have

$$P_U(\boldsymbol{q}|f=1) = \frac{P_U(f=1|\boldsymbol{q}) P_U(\boldsymbol{q})}{P_U(f=1)} = P_U(\boldsymbol{q}) \tag{46}$$

where the second equality is due to the independence between the flag f and the queue status (similar to the proof of Theorem 4 and see also the discussion in [5, Sec. 3.3.2]).

Note that the number of arrival transitions between two successive departure transitions yields the geometric distribution as given in (18). $P_U(\boldsymbol{q}|f=1)$ is the stationary distribution of the original Markov chain measured before the transmission moment in each slot. Lemma 5 then follows from (46).

□

Appendix F
Proof of Lemma 6

Define the parameter Λ, the joint distribution $P_a(\Delta\boldsymbol{q})$ as in Lemma 5, and the departure transition $P_d(\boldsymbol{q}_i|\boldsymbol{q}_j)$ as in the proof of Lemma 5.

Now, we construct a new discrete-time Markov chain as follows. We associate an "arrival/departure" flag f to each transition of the new Markov chain. The f flag is determined before the corresponding transitions; $f = 0$ with probability $\frac{\Lambda}{1+\Lambda}$ and $f = 1$ with probability $\frac{1}{1+\Lambda}$. Assume the f flag is generated independently among different slots and is also independent to all other events. In each slot, a virtual transmission flag vector $\boldsymbol{t}^{(v)}$ is generated assuming all terminals have packets to transmit. If $t_i^{(v)} = 1$ and the real queue of terminal i is not empty, then the real transmission flag $t_i = 1$, otherwise $t_i = 0$. We then associate a flag g_i to terminal i, for all i. We let $g_i = 0$ if the queue of terminal i is not empty or $b_i t_i^{(v)} = 0$; and let $g_i = 1$ otherwise.

Given the queue status $\boldsymbol{q}_j$, each transition can either be an "arrival" transition, a "departure" transition, or an "idle" transition, depending on the associated flags f and $\boldsymbol{g}_{\mathcal{S}}$. If $f = 0$ and $\boldsymbol{g}_{\mathcal{S}} = \boldsymbol{0}$, the transition is an "arrival" transition and the queue status changes from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ according to the arrival transition probability $P_a(\boldsymbol{q}_i - \boldsymbol{q}_j)$. If $f = 1$ and $\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}$, the corresponding transition is a "departure" transition and the queue status changes from $\boldsymbol{q}_j$ to $\boldsymbol{q}_i$ according to the departure transition probability $P_d(\boldsymbol{q}_i|\boldsymbol{q}_j)$. If $\boldsymbol{g}_\mathcal{S} \neq \boldsymbol{0}$, however, the transition is an "idle" where the queue status stays at $\boldsymbol{q}_j$ with probability one.

We first show the constructed Markov chain is monotonic. Consider two copies of the constructed Markov chain, $\boldsymbol{q}^{(1)}(n)$ and $\boldsymbol{q}^{(2)}(n)$. We couple the two Markov chains as follows. Assume the virtual transmission flag vectors $\boldsymbol{t}^{(v)}$ of both Markov chains are identical, and are generated according to the description in the beginning of this proof. We generate the transmission flag vector $\boldsymbol{t}^{(1)}$ and let $t_i^{(1)} = 1$ if and only if $t_i^{(v)} = 1$ and $q_i^{(1)} > 0$. The transmission flag vector $\boldsymbol{t}^{(2)}$ is generated similarly. Let the flag vectors $\boldsymbol{b}^{(1)} = \boldsymbol{b}^{(2)}$, and calculate the vectors $\boldsymbol{v}_\mathcal{S}^{(1)}$ and $\boldsymbol{v}_\mathcal{S}^{(2)}$, respectively. We assume other parameters and channel statistics of the two Markov chains are identical.

Under the constraint of (19), the flag vector $\boldsymbol{b}$ can be interpreted as the "blocking flag" in the sense that if, in a particular slot, a packet from terminal i is received, $\boldsymbol{v}_{\overline{\{i\}}} = \boldsymbol{0}$ must be true. Here $\overline{\{i\}}$ is the group of all terminals except terminal i. In other words, to receive a packet from terminal i, other terminal either does not transmit at all, or transmits without the "blocking flag."

Suppose at time slot n, we have $\boldsymbol{q}^{(2)}(n) \leq \boldsymbol{q}^{(1)}(n)$. According to the coupling, if the Markov chain $\boldsymbol{q}^{(2)}(n)$ takes an "arrival" or a "departure" transition, the same type of transition must also be taken by the Markov chain $\boldsymbol{q}^{(2)}(n)$. In these cases, since the MPR channel is standard, it is easy to see that $\boldsymbol{q}^{(2)}(n+1) \leq \boldsymbol{q}^{(1)}(n+1)$ is satisfied after the transition. If Markov chain $\boldsymbol{q}^{(1)}(n)$ takes an "arrival" transition while Markov chain $\boldsymbol{q}^{(2)}(n)$ takes an "idle" transition, $\boldsymbol{q}^{(2)}(n+1) \leq \boldsymbol{q}^{(1)}(n+1)$ certainly holds after the transition. Now, consider the situation when Markov chain $\boldsymbol{q}^{(1)}(n)$ takes a "departure" transition while Markov chain $\boldsymbol{q}^{(2)}(n)$ takes an "idle" transition. According to the coupling, there must be a terminal $i \in \mathcal{S}$, such that

$$v_i^{(1)} = 1, \qquad v_i^{(2)} = 0. \tag{47}$$

Since $b_i^{(1)} = b_i^{(2)}$, (47) implies that

$$t_i^{(1)} = 1, \qquad t_i^{(2)} = 0, \qquad b_i^{(1)} = b_i^{(2)} = 1. \tag{48}$$

Since both $t_i^{(1)}$ and $t_i^{(2)}$ are generated from $t_i^{(v)}$ and the corresponding queue status, (48) consequently implies

$$q_i^{(1)} > 0, \qquad q_i^{(2)} = 0. \tag{49}$$

Since $b_i^{(1)} = 1$, which means terminal i transmits a packet with the blocking flag raised, no packets from other terminals can be received successfully. Together with (49), we can see $\boldsymbol{q}^{(2)}(n+1) \leq \boldsymbol{q}^{(1)}(n+1)$ still holds after the transition.

From the preceding analysis, we conclude that the constructed continuous-time Markov chain is monotonic. It is easy to see that if the original system is stable, the constructed Markov chain is also stable; since if the queue of any terminal i is nonempty with probability 1, then $g_i = 0$ with probability 1, and this leads back to the Markov chain constructed in the proof of Lemma 5. Since the constructed Markov chain has only up or down transitions, the stationary distribution of the queues has positive correlation, according to Lemma 4.

Now, let us consider the stationary distribution of the queues of the constructed Markov chain, denoted by $\tilde{U}$. For any state $\boldsymbol{q}_i$, the stationary probability $P_{\tilde{U}}(\boldsymbol{q}_i)$ satisfies

$$\begin{aligned} P_{\tilde{U}}(\boldsymbol{q}_i) &= \frac{\Lambda}{1+\Lambda} \sum_{\boldsymbol{q}_j, \boldsymbol{q}_j \leq \boldsymbol{q}_i} P_a(\boldsymbol{q}_i - \boldsymbol{q}_j) P_{\tilde{U}}(\boldsymbol{q}_j) P(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q}_j) \\ &\quad + \frac{1}{1+\Lambda} \sum_{\boldsymbol{q}_j, \boldsymbol{q}_i \leq \boldsymbol{q}_j} P_d(\boldsymbol{q}_i|\boldsymbol{q}_j) P_{\tilde{U}}(\boldsymbol{q}_j) P(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q}_j) \\ &\quad + P_{\tilde{U}}(\boldsymbol{q}_i) P(\boldsymbol{g}_\mathcal{S} \neq \boldsymbol{0}|\boldsymbol{q}_i). \end{aligned} \tag{50}$$

Consequently, we have

$$\begin{aligned} P_{\tilde{U}}(\boldsymbol{q}_i|\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) &= \frac{\Lambda}{1+\Lambda} \sum_{\boldsymbol{q}_j, \boldsymbol{q}_j \leq \boldsymbol{q}_i} P_a(\boldsymbol{q}_i - \boldsymbol{q}_j) P_{\tilde{U}}(\boldsymbol{q}_j|\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) \\ &\quad + \frac{1}{1+\Lambda} \sum_{\boldsymbol{q}_j, \boldsymbol{q}_i \leq \boldsymbol{q}_j} P_d(\boldsymbol{q}_i|\boldsymbol{q}_j) P_{\tilde{U}}(\boldsymbol{q}_j|\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}). \end{aligned} \tag{51}$$

Comparing (51) with (45), we conclude that

$$P_{\tilde{U}}(\boldsymbol{q}|\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) = P_U(\boldsymbol{q}) \tag{52}$$

where U is the stationary distribution of the Markov chain constructed in the proof of Lemma 5, and is also the stationary distribution of the original system.

From the construction of flag vectors $\boldsymbol{g}$ and $\boldsymbol{v}$, it is easy to verify the following equality:

$$P(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q}) P(\boldsymbol{v}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q}) = P(b_i t_i^{(v)} = 0,\ \forall i \in \mathcal{S}) \tag{53}$$

where the right-hand side of (53) is independent of the queue status $\boldsymbol{q}$. Combining (52) and (53), we obtain

$$\begin{aligned} P_U(\boldsymbol{q}) &= P_{\tilde{U}}(\boldsymbol{q}|\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) \\ &= \frac{P_{\tilde{U}}(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q}) P_{\tilde{U}}(\boldsymbol{q})}{P_{\tilde{U}}(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0})} \\ &= \frac{P(b_i t_i^{(v)} = 0,\ \forall i \in \mathcal{S})}{P_{\tilde{U}}(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0})} \frac{P_{\tilde{U}}(\boldsymbol{q})}{P(\boldsymbol{v}_\mathcal{S} = \boldsymbol{0}|\boldsymbol{q})}. \end{aligned} \tag{54}$$

This gives

$$P_U(\boldsymbol{q}|\boldsymbol{v}_\mathcal{S} = \boldsymbol{0}) = \frac{P(b_i t_i^{(v)} = 0, \forall i \in \mathcal{S})}{P_{\tilde{U}}(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) P_U(\boldsymbol{v}_\mathcal{S} = \boldsymbol{0})} P_{\tilde{U}}(\boldsymbol{q}). \tag{55}$$

Given the distributions U and $\tilde{U}$

$$\frac{P(b_i t_i^{(v)} = 0,\ \forall i \in \mathcal{S})}{P_{\tilde{U}}(\boldsymbol{g}_\mathcal{S} = \boldsymbol{0}) P_U(\boldsymbol{v}_\mathcal{S} = \boldsymbol{0})}$$

is a constant not depending on $\boldsymbol{q}$. Since both $P_U(\boldsymbol{q}|\boldsymbol{v}_\mathcal{S} = \boldsymbol{0})$ and $P_{\tilde{U}}(\boldsymbol{q})$ are probability mass functions with respective to $\boldsymbol{q}$, (55) implies

$$P_{\tilde{U}}(\boldsymbol{q}) = P_U(\boldsymbol{q}|\boldsymbol{v}_\mathcal{S} = \boldsymbol{0}). \tag{56}$$

Since $P_{\tilde{U}}(\boldsymbol{q})$ has positive correlation, according to Lemma 4, $P_U(\boldsymbol{q}|\boldsymbol{v}_\mathcal{S} = \boldsymbol{0})$ also has positive correlation. □

APPENDIX G
PROOF OF LEMMA 7

The key idea of the proof is to construct a new MPR channel, such that the stability region of the original system is contained in the stability region of the new system. Meanwhile, we construct the new MPR channel so that the parameters have a nice structure, which allows us to apply the framework presented in

[21] and obtain the closed-form expression of the stability region.

The new system is constructed as follows. We first define a "channel availability" flag $c_f \in \{0,1\}$. In each slot, c_f is generated with $P(c_f = 1) = 1 - \mathsf{q}_0$ and $P(c_f = 0) = \mathsf{q}_0$, independent to all other events. Assume $\boldsymbol{p}_b \in P_B$ is an arbitrary probability vector in P_B. Let the "blocking flag" vector $\boldsymbol{b}$ be generated with b_i and b_j being independent for all i, j, and $P(b_i = 1) = p_{bi}$. In any slot, if terminals in $\mathcal{S}$ transmit packets and terminals in $\bar{\mathcal{S}}$ do not transmit, for an arbitrary terminal $i \in \mathcal{S}$, the packet from i is received if and only if the channel is available ($c_f = 1$) and it sees no other packets with blocking flag ($b_j = 0$, $j \in \mathcal{S}$, $j \neq i$). With the definition of c_f and the constraints (21), given the transmitter set $\mathcal{S}$, the packet reception of the original system is stochastically dominated by that of the new system. Hence, the stability region of the original system is contained in the stability of the new system.

For the new system, suppose $\tilde{\boldsymbol{\lambda}}$ is a stable packet arrival rate vector. According to Lemma 6, the conditional stationary distribution of the queue status $U(\boldsymbol{q}|\boldsymbol{v}_{\mathcal{S}} = \boldsymbol{0})$ has positive correlation. Therefore, for any $k > 1$, we have

$$P_U(\overline{v}_1\overline{v}_k|\overline{v}_2\ldots\overline{v}_{k-1}) \geq P_U(\overline{v}_1|\overline{v}_2\ldots\overline{v}_{k-1})P_U(\overline{v}_k|\overline{v}_2\ldots\overline{v}_{k-1}) \quad (57)$$

where we use $\overline{v}_i$ to denote the event of $v_i = 0$. Follow the discussions presented by Anantharam in [21, Sec. IV], (57) implies

$$P_U(\overline{v}_1\ldots\overline{v}_N)^{N-1} \geq \prod_{i=1}^{N} P_U(\overline{v}_1\ldots\overline{v}_{i-1}\overline{v}_{i+1}\ldots\overline{v}_N). \quad (58)$$

Since

$$P_U(\overline{v}_1\ldots\overline{v}_{i-1}\overline{v}_{i+1}\ldots\overline{v}_N) = P_U(\overline{v}_1\ldots\overline{v}_N) + P_U(\overline{v}_1\ldots v_i\ldots\overline{v}_N) \quad (59)$$

and

$$P_U(\overline{v}_1\ldots v_i\ldots\overline{v}_N) = \frac{P_U(\overline{v}_1\ldots t_i\ldots\overline{v}_N c_f)p_{bi}}{1-\mathsf{q}_0} = \frac{\tilde{\lambda}_i p_{bi}}{1-\mathsf{q}_0}. \quad (60)$$

Defining $x = P_U(\overline{v}_1\ldots\overline{v}_N)$, we obtain from (58) the following inequality:

$$x^{N-1} \geq \prod_{i=1}^{N}\left(x + \frac{\tilde{\lambda}_i p_{bi}}{1-\mathsf{q}_0}\right). \quad (61)$$

Define

$$\tilde{p}_i = \frac{\frac{\tilde{\lambda}_i}{1-\mathsf{q}_0}}{x + \frac{\tilde{\lambda}_i p_{bi}}{1-\mathsf{q}_0}}. \quad (62)$$

We obtain from (61)

$$\tilde{\lambda}_i \leq (1-\mathsf{q}_0)\tilde{p}_i \prod_{j\neq i}^{N}(1-\tilde{p}_j p_{bj}). \quad (63)$$

Note that

$$\begin{aligned}\tilde{\lambda}_i &\leq P_U(\overline{v}_1\ldots t_i\ldots\overline{v}_N c_f)\\ &= \frac{1}{1-p_{bi}}P_U(\overline{v}_1\ldots t_i\overline{b}_i\ldots\overline{v}_N c_f)\\ &\leq \frac{1}{1-p_{bi}}P_U(\overline{v}_1\ldots\overline{v}_i\ldots\overline{v}_N c_f)\\ &= \frac{(1-\mathsf{q}_0)x}{1-p_{bi}}.\end{aligned} \quad (64)$$

We get

$$\tilde{p}_i = \frac{\frac{\tilde{\lambda}_i}{1-\mathsf{q}_0}}{x + \frac{\tilde{\lambda}_i p_{bi}}{1-\mathsf{q}_0}} \leq 1 \quad (65)$$

and hence $\tilde{p}_i$ can be interpreted as a probability. From (63), and since $\mathcal{C}(\boldsymbol{p}_b)$ is "coordinate convex" [21], we conclude that the packet arrival rate vector $\tilde{\boldsymbol{\lambda}}$ is located inside the region $\mathcal{C}(\boldsymbol{p}_b)$. Since the stability region of the original system is contained in the stability region of the new system, and $\boldsymbol{p}_b$ is arbitrary, the packet arrival rate vector $\boldsymbol{\lambda}$ of the original system must be located inside $\hat{\mathcal{C}}_S$. □

APPENDIX H
PROOF OF LEMMA 8

Construct a dominant system where terminal i transmits dummy packet with probability p_i when its buffer is empty. If the MPR channel is standard, such a dummy packet transmission decreases the reception probability of packets from other terminals. Hence, according to the analysis in [7], [9], the distribution of the queues of the original system is stochastically dominated by the dominant system. Since given the queue status $\boldsymbol{q}$, the channel availability probability $P(a_i = 1|\boldsymbol{q})$ is an decreasing function in $\boldsymbol{q}$, denote $P_{\tilde{U}}(a_i = 1)$ as the stationary probability of $a_i = 1$ in the dominant system, we have

$$P_{\tilde{U}}(a_i = 1) \leq P_U(a_i = 1). \quad (66)$$

Note that the queue of terminal i must be stable in the dominant system, since otherwise, once the queue of terminal i builds up, the probability of q_i back to 0 becomes zero, and therefore contradicts the stability assumption of the original system. Consequently, we get

$$\lambda_i \leq p_i P_{\tilde{U}}(a_i = 1) \leq p_i P_U(a_i = 1). \quad (67)$$

On the other hand, if $P_U(a_i = 1)$ is the stationary probability of $a_i = 1$, and

$$\lambda_i < p_i P_U(a_i = 1) \quad (68)$$

then the queue of terminal i must be stable. Since if q_i is not stable, the packet transmission probability of terminal i becomes p_i and the instability of q_i contradicts (68). □

APPENDIX I
PROOF OF THEOREM 2

Denote the original ALOHA system by A_0, and let $\boldsymbol{p}^{(A_0)}$ be the corresponding transmission probability vector. Let $\boldsymbol{\lambda}$ be the packet arrival rate vector. Since the system is stable, we have, for all i

$$\lambda_i \leq p_i^{(A_0)} P_{A_0}(a_i = 1). \quad (69)$$

Now, construct a new system A_1. $\forall i$, let the transmission probability vector of terminal i in A_1 be

$$p_i^{(A_1)} = \frac{\lambda_i}{P_{A_0}(a_i = 1)}. \quad (70)$$

According to the conjectured sensitivity monotonicity property, we have

$$p_i^{(A_1)} P_{A_1}(a_i = 1) \geq P_{A_0}(a_i = 1) \frac{\lambda_i}{P_{A_0}(a_i = 1)} = \lambda_i. \quad (71)$$

We continue constructing a new system A_n from A_{n-1} by letting the transmission probability vector in A_n be

$$p_i^{(A_n)} = \frac{\lambda_i}{P_{A_{n-1}}(a_i = 1)}. \quad (72)$$

It is easy to see that

$$p_i^{(A_n)} P_{A_n}(a_i = 1) \geq \lambda_i, \qquad \forall n. \quad (73)$$

Since $p_i^{(A_n)}$ is monotonically decreasing in n, asymptotically, we have

$$p_i^{(A_\infty)} P_{A_\infty}(a_i = 1) = \lambda_i. \quad (74)$$

This implies, in system A_∞

$$P_{A_\infty}(q_i > 0) = 1, \qquad \forall i. \quad (75)$$

Consequently, packet transmissions from different terminals are independent, and hence, $p_i^{(A_\infty)} P_{A_\infty}(a_i = 1)$ is given by

$$p_i^{(A_\infty)} P_{A_\infty}(a_i = 1) = \lambda_i = T_i(\boldsymbol{p}^{(A_\infty)}). \quad (76)$$

Therefore, $\boldsymbol{\lambda}$ is inside the region $\mathcal{C}_S$. Since $\mathcal{C}_S$ is also an inner bound to the closure of stability region [16], $\mathcal{C}_S = \mathcal{C}_T$ must be the closure of the stability region. □

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