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A Fast Optimal Algorithm for CWMA Multiuser Detection

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Abstract — **A fast optimal algorithm based on branch and bound method is proposed for the joint detection of binary symbols of K users in a synchronous correlated waveform multiple-access (CWMA) channel with Gaussian noise. We consider the detection problem as one of optimizing a quadratic objective function with binary constraints on decision variables. The average computational cost of finding the optimal solution is $O(K^2)$, which is of the same order as those of the familiar sub-optimal algorithms. Simulation results show that the optimal solution can provide 3 orders or more of magnitude improvement in the probability of error.**

I. Introduction

Due to the problem of Interuser Interference (IUI) in many multiuser communication systems, multiuser detection for the symbol-synchronous Gaussian correlated waveform multiple-access (CWMA) channel has received considerable attention over the past ten years. Because of the discrete nature of the signal, all existing detectors need to perform a projection to satisfy the integrality constraints. Such a projection can cause significant errors.

Based on the idea of successive cancellation, a systematic Decision Feedback Detection (DFD) approach was given in [1]. Generally, DFD methods can give a significant improvement in probability of error when compared with traditional methods, while maintaining the same computational complexity of $O(K^2)$. However, computer simulations show that there is still a large gap between the probability of error of DFD outputs and that of the optimal solution. In this paper, a fast optimal algorithm based on the idea of branch and bound is proposed. The Minimum Mean Square Error (MMSE) method is used to provide a tight lower bound for each branch. Theoretical analysis shows that the number of multiplications for lower bound estimation is linear in the number of unassigned users. In addition, the tight MMSE lower bound helps greatly in cutting the number of branches, thereby decreasing the number of sub-problems. Computer simulations for a 10-user system show that the magnitude improvement in the probability of eror can be 3 orders or more when compared with the D-DFD method [1], while the average number of multiplications is around $2K^2$. Furthermore, theoretical discussion is given to show that the DFD method is in fact an order-one approximation to the optimal solution.

This paper is organized as follows. The synchronous multiuser detection problem formulation and existing solution techniques are discussed in section 2. In section 3, the fast optimal algorithm to reduce the number of sub-problems is presented, and theoretical analysis of the computational complexity is given. The idea of order-one sub-optimal approximation is proposed in section 4 and is proved to be the DFD method. Simulation results and comparative analyses are provided in section 5. The paper concludes with a summary in section 6.

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II. Problem Formulation and Existing Methods

A discrete-time equivalent model for the matched-filter outputs at the receiver of a CWMA channel is given by the K-length vector [1]

$$y = Hb + N \tag{1}$$

where $b \in \{-1,+1\}^K$ denotes the K-length vector of bits transmitted by the K active users. Here $H = W^{\frac{1}{2}} R W^{\frac{1}{2}}$ is a nonnegative definite signature waveform correlation matrix, R is the symmetric normalized correlation matrix with unit diagonal elements, W is a diagonal matrix whose k-th diagonal element, w_k is the received signal energy per bit of the k-th user, and N is a real-valued zero-mean Gaussian random vector with a covariance matrix $\sigma^2 H$. It has been shown that this model holds for both baseband [2] and passband [1] channels with additive Gaussian noise.

When all the user signals are equally probable, the optimal solution of (1) is the output of a Maximum Likelihood (ML) detector [1] (also a Maximum A Posteriori (MAP) receiver in this case)

$$\phi_{ML} : \hat{b} = \arg \min_{b\in\{-1,+1\}^K} b^T H b - 2y^T b \tag{2}$$

The ML detector has the property that it minimizes, among all detectors, the probability that not all users' decisions are correct. Usually, ϕ_{ML} is considered NP-hard and exponentially complex to implement; the focus is then on developing easily implementable and effective multiuser detectors.

The MMSE-based solution of conventional decorrelation detector [2]

$$\phi_D : \hat{b} = \arg \min_{b\in\{-1,+1\}^K} \left\| b - H^{-1} y \right\|_2^2 \tag{3}$$

is found in two steps. First, the unconstrained solution $\tilde{b} = H^{-1}y$ is computed. This is then projected onto the constraint set via: $\hat{b}_i = \text{sign}\left(\tilde{b}_i\right)$.

The DFD method based on the decorrelation detector is described in [1] by

$$\phi_{D-DFD} : \hat{b} = P\tilde{b},\ \tilde{b}_i = \text{sign}\left(\sum_{j=1}^{K} F_{ij} P y_j - \sum_{j=1}^{i-1} B_{ij} \tilde{b}_j\right) \tag{4}$$

where $F = \text{U}\left([PHP]^{-1}\right)$, $B = \text{L}(FPHP)$. Here, $\text{U}(\cdot)$ represents the upper triangular part of a matrix and $\text{L}(\cdot)$ represents the strictly lower triangular part of a matrix. P is a permutation matrix, such that, for the Cholesky decomposition of $PHP = \tilde{L}\tilde{L}^T$, we have [1]

$$\tilde{L}_{11}^2 \geq \tilde{L}_{22}^2 \geq ... \geq \tilde{L}_{KK}^2 \tag{5}$$

Without loss of generality, in the rest of this paper, we assume $\tilde{L}\tilde{L}^T = H$ and $\tilde{L}$ satisfies (5).
The number of multiplications for the above two algorithms is exactly K^2. In general, D-DFD methods perform well and can provide two to three orders of improvement in magnitude of probability of error when compared with conventional linear detectors. However, the output of D-DFD is still a suboptimal solution. Simulation results show that, in most of the cases, there still exists a substantial gap in performance between the D-DFD and the optimal solution.

III. Optimal Algorithm Based on Branch and Bound

The idea of using a branch and bound method in solving optimization problems is already well known [3]. However, the tradeoff between a tight lower bound and a lower bound with less computational requirements is common in most of the problems [4]. In this paper, the fast optimal algorithm uses a tight MMSE lower bound and we find an iterative way to update the lower bound such that the computational cost for each estimation is linear in the number of the rest of unassigned users.

We will first give a useful proposition, which is the key idea of the fast optimal algorithm in this paper.
Proposition 1: Suppose $H^{-1} = LL^T$ is the Cholesky decomposition of H^{-1}, and $\hat{L} = L^{-1}$ is the inverse of L. Suppose

$$H = \begin{bmatrix} H_{(K-n),(K-n)} & H_{n,(K-n)}^T \\ H_{n,(K-n)} & H_{n,n} \end{bmatrix} \tag{6}$$

is the partition of H on the $(K-n)$th diagonal element, and

$$L = \begin{bmatrix} L_{(K-n),(K-n)} & 0 \\ L_{n,(K-n)} & L_{n,n} \end{bmatrix}$$
$$\hat{L} = \begin{bmatrix} \hat{L}_{(K-n),(K-n)} & 0 \\ \hat{L}_{n,(K-n)} & \hat{L}_{n,n} \end{bmatrix} \tag{7}$$

are the corresponding partitions for L and $\hat{L}$, respectively. Then, we have the following results:

$$\hat{L}_{(K-n),(K-n)} = L_{(K-n),(K-n)}^{-1}$$
$$\hat{L}_{n,n} = L_{n,n}^{-1} \tag{8}$$

$$H_{n,n}^{-1} = L_{n,n}L_{n,n}^T$$
$$H_{n,(K-n)} = \left(L_{n,n}^{-1}\right)^T \hat{L}_{n,(K-n)} \tag{9}$$

Proof: Since $L\hat{L} = I$, we have

$$\begin{bmatrix} L_{(K-n),(K-n)}\hat{L}_{(K-n),(K-n)} & 0 \\ L_{n,(K-n)}\hat{L}_{(K-n),(K-n)} + L_{n,n}\hat{L}_{n,(K-n)} & L_{n,n}\hat{L}_{n,n} \end{bmatrix} = I \tag{10}$$

and immediately gives (8). In addition, since $H^{-1} = LL^T$, we have $H = \hat{L}^T\hat{L}$, which implies

$$H = \begin{bmatrix} H_{(K-n),(K-n)} & \hat{L}_{n,(K-n)}^T\hat{L}_{n,n} \\ \hat{L}_{n,n}^T\hat{L}_{n,(K-n)} & \hat{L}_{n,n}^T\hat{L}_{n,n} \end{bmatrix} \tag{11}$$

Hence (9) can be easily obtained.

In minimizing the objective function (2), suppose, on an arbitrary branch, b, y and H in (2) can be partitioned as

$$b = \begin{bmatrix} b_{(K-n)} & b_n \end{bmatrix}^T$$
$$y = \begin{bmatrix} y_{(K-n)} & y_n \end{bmatrix}^T$$
$$H = \begin{bmatrix} H_{(K-n),(K-n)} & H_{n,(K-n)}^T \\ H_{n,(K-n)} & H_{n,n} \end{bmatrix} \tag{12}$$

where $b_{(K-n)}$ has already been fixed by the branch. The objective function of the user-expurgated channel with respect to b can then be written as

$$f(b) = b_n^T H_{n,n} b_n - 2\left(y_n - H_{n,(K-n)}b_{(K-n)}\right)^T b_n$$
$$+ b_{(K-n)}^T H_{(K-n),(K-n)} b_{(K-n)} - 2y_{(K-n)}^T b_{(K-n)} \tag{13}$$

It is easy to see that the unconstrained MMSE solution for (13) is

$$\tilde{b}_n = H_{n,n}^{-1}\left(y_n - H_{n,(K-n)}b_{(K-n)}\right) \tag{14}$$

Hence, by using the result of proposition 1, we have

$$\min_{b_n} f(b) = g_n\left(b_{(K-n)}\right)$$
$$= -\left(\xi_n - \psi_n\right)^T\left(\xi_n - \psi_n\right) + \mu_n - 2\nu_n \tag{15}$$

where

$$\xi_n = L_{n,n}^T y_n$$
$$\psi_n = \hat{L}_{n,(K-n)}b_{(K-n)}$$
$$\mu_n = b_{(K-n)}^T H_{(K-n),(K-n)} b_{(K-n)} \text{and}$$
$$\nu_n = y_{(K-n)}^T b_{(K-n)} \tag{16}$$

Initially, we have $\xi_K = L^T y$, $\psi_K = 0$, $\mu_K = 0$, $\nu_K = 0$. With iterative update, the following formulae (17)~(19) enable us to obtain ξ_n, ψ_n, μ_n, ν_n from ξ_{n+1}, ψ_{n+1}, μ_{n+1}, ν_{n+1} with $2n+4$ multiplications.

$$\xi_{n+1} = L_{(n+1),(n+1)}^T y_{(n+1)} = \begin{bmatrix} [\xi_{n+1}]_1 \\ \xi_n \end{bmatrix} \tag{17}$$

Here $[\xi_{n+1}]_1$ represents the first component of ξ_{n+1}. Thus, equation (17) shows that we can obtain ξ_n from ξ_{n+1} directly.

Similarly,

$$\psi_n = [\psi_{n+1}]_{2:n+1} + \left[\hat{L}_{n,(K-n)}\right]_{:,K-n}\left[b_{(K-n)}\right]_{K-n} \tag{18}$$

where $[\psi_{n+1}]_{2:n+1}$ represents the sub-vector of ψ_{n+1} with the first component subtracted out from ψ_{n+1}. In (18), $\left[\hat{L}_{n,(K-n)}\right]_{:,K-n}$ represents the $(K-n)$th column of $\hat{L}_{n,(K-n)}$. This implies that n multiplications are needed to get ψ_n from ψ_{n+1}. To obtain μ_n, we have

$$\mu_n = \mu_{n+1} + 2\left[b_{K-n}\right]_{K-n}\left[\hat{L}_{(n+1),(n+1)}\right]_{1,:}\psi_{n+1}$$
$$+ \left[H_{n+1,n+1}\right]_{1,1}\left(\left[b_{K-n}\right]_{K-n}\right)^2 \tag{19}$$

Hence, we need $n+1$ multiplications to obtain $\left[\hat{L}_{(n+1),(n+1)}\right]_{1,:}\psi_{n+1}$, and another 2 multiplications to determine μ_n. Finally, we need 1 multiplication to evaluate ν_n as

$$\nu_n = \nu_{n+1} + \left[y_{(K-n)}\right]_{K-n}\left[b_{(K-n)}\right]_{K-n} \tag{20}$$

An extra n multiplications are needed to obtain $g_n\left(b_{(K-n)}\right)$. Hence the overall number of multiplications needed is $3n+4$. Further simplification is possible for μ_n when $n > \frac{K}{2}$, which we will omit for the sake of brevity.

In addition to the iterative update of lower bounds, we first sort the users according to (4) and (5), which is, in most cases, the optimal detection sequence for the DFD method [1]. Similar to the proof in [1], it is easy to see that this is also the best detection sequence in terms of computational complexity for the branch and bound algorithm.

IV. Sub-Optimal Alternative and The DFD Method

Although the proposed branch and bound method works very fast on average, the original problem (2) has been proved to be NP-hard in general, hence there is no guarantee of getting the result in polynomial time. In this section, we will focus on the sub-optimal alternative when strict limitations on computational cost exist.

Proposition 2: In the optimal algorithm, when we branch, we choose that with the smaller lower bound and push the other one to the stack. By doing this, the first feasible solution we obtain will be exactly the output of the D-DFD method.

Proof: Suppose for each branch, before branching down according to (13), we update the data as,

$$\begin{aligned}\tilde{y} &= y_n - H_{n,(K-n)} b_{(K-n)}\\ \tilde{H} &= H_{n,n}\\ \tilde{b} &= b_n\end{aligned} \tag{21}$$

The optimization within this branch will be equivalent to

$$\min_{\tilde{b}} f(b) = \min_{\tilde{b}} \left(\tilde{y}^T \tilde{H} \tilde{y} - 2\tilde{y}^T \tilde{b}\right) \tag{22}$$

Then, for one level branching, we denote $\tilde{b}$, $\tilde{y}$, $\tilde{H}$ by

$$\begin{aligned}\tilde{b} &= \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_{n-1} \end{bmatrix}\\ \tilde{y} &= \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_{n-1} \end{bmatrix}\\ \tilde{H} &= \begin{bmatrix} \tilde{H}_{1,1} & \tilde{H}_{(n-1),1} \\ \tilde{H}_{(n-1),1} & \tilde{H}_{1,1} \end{bmatrix}\end{aligned} \tag{23}$$

Note here, since we just branch down by one level, $\tilde{b}_1$, $\tilde{H}_{1,1}$ and $\tilde{y}_1$ are all scalars. The MMSE lower bound of the sub-branch is a function of $\tilde{b}_1$. Considering $\tilde{b}_1 \in \{-1, 1\}$, from (15), we can easily see that,

$$\arg\min_{\tilde{b}_1} \min_{\tilde{b}_{(n-1)}} f\left(\tilde{b}\right) = \text{sign}\left[\tilde{y}_1 - \tilde{y}^T_{(n-1)} M_{OPT}\right] \tag{24}$$

where

$$M_{OPT} = L^{-1}_{(n-1),(n-1)} \left(\hat{L}_{(n-1),(K-n+1)}\right)_{:,(K-n+1)} \tag{25}$$

From the discussion of [1], we know the decision for D-DFD is

$$\begin{aligned}\tilde{b}_{1DFD} &= \text{sign}\left[\left(\tilde{H}^{-1}\right)\tilde{y}\right]_1\\ &= \text{sign}\left[[\tilde{y}]_1 + M^T_{DFD}\tilde{y}_{(n-1)}\right]\end{aligned} \tag{26}$$

where

$$M^T_{DFD} = \left[L_{n,n}\right]^{-1}_{11} \left[L_{(n-1),(K-n+1)}\right]_{:,(K-n+1)} \tag{27}$$

Using $L_{n,n}\tilde{L}_{n,n} = I$, we obtain

$$M_{DFD} = M_{OPT} \tag{28}$$

Since the number of multiplications to obtain the first feasible solution is $1.5K^2 + 2.5K - 1$, when the strict computational limitation is above this number, we can simply pick the current-best solution as a sub-optimal alternative.

V. Simulation Results

In this section, we compare the error performances and complexities of Decorrelation method, optimal D-DFD method, and the fast optimal algorithm. A K-user CWMA channel is considered ($K = 10$ in Figure 1). The signature waveform correlation matrix H is generated as follows: First, the signature correlation matrix R is found by computing the correlation between randomly generated signature sequences of 14-bits length each. The condition number of R is chosen to be greater than 20. Second, the signal energies (i.e., the elements of matrix W) are generated within the range $[1, 3.0]$, and H is calculated as $H = W^{\frac{1}{2}} R W^{\frac{1}{2}}$. The SNR is chosen so that the probability of error (calculated based on importance sampling of 10000 Monte-Carlo runs) is similar to that observed in real applications.

Figure 1 shows the probability of error for group detection. A sub-optimal alternative with a strict $5K^2$ multiplication limit is also presented. The number of multiplications needed for optimal algorithm under different SNRs as a function of number of users are shown in Figure 2. In most of the cases when noise is small, we can get the optimal solution in $O\left(2K^2\right)$ multiplications.

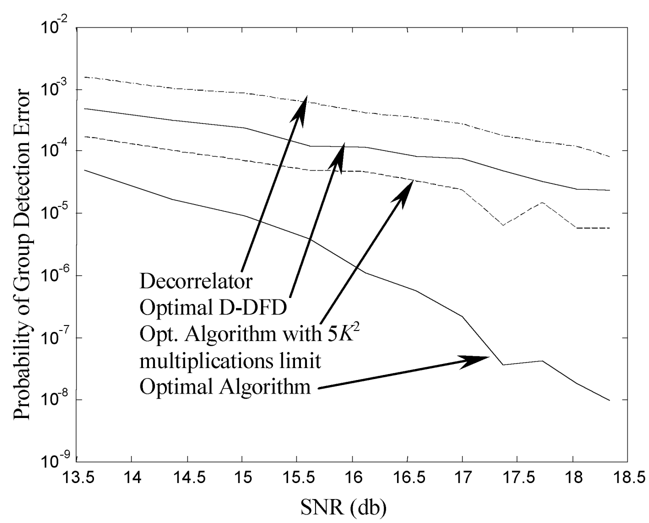

Figure 1: Performance of various methods ($K = 10$)

VI. Conclusion

Typically, the optimal solution to multi-user detection is regarded as NP-hard and unlikely to be implemented. However, the algorithm proposed in this paper shows that there does

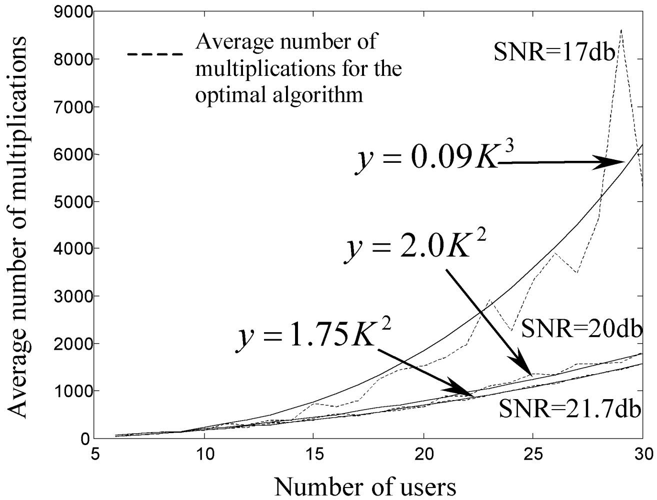

Figure 2: Number of multiplications (The numbers of multiplications for the Decorrelation and the D-DFD method are exactly K^2)

exist a fast implementation of the optimal algorithm, which on average, has the same order of computational cost as those of conventional methods. All the above results can be easily extended to integral signals instead of binary ones.

References

[1] M. K. Varanasi, "Decision feedback multiuser detection: a systematic approach", *IEEE Trans. Inform. Theory*, vol. 45, pp. 219–240, Jan. 1999.

[2] R. Lupas and S. Verdu, "Linear multiuser detectors for synchronous code-devision multiple-access channels", *IEEE Trans. Inform. Theory*, vol. 35, pp. 123–136, Jan. 1989.

[3] C. Schlegel and L. Wei, "A simple way to compute the minimum distance in multiuser CDMA systems", *IEEE Trans. Communications*, vol. 45, pp. 532–535, May 1997.

[4] P.M. Pardalos, G.P. Rodgers, "Computational aspects of a branch and bound algorithm for quadratic zero-one programming", *Computing* vol. 45, pp. 131–144, 1990.