Optimal Grouping Algorithm for a Group Decision Feedback Detector in Synchronous CDMA Communications

J. Luo, K. Pattipati, P. Willett, G. Levchuk

Abstract— **The Group Decision Feedback (GDF) detector is studied in this paper. The computational complexity of a GDF detector is exponential in the largest size of the groups. Given the maximum group size, a grouping algorithm is proposed. It is shown that the proposed grouping algorithm maximizes the Asymptotic Symmetric Energy (ASE) of the multiuser detection system. Furthermore, based on a set of lower bounds on Asymptotic Group Symmetric Energy (AGSE) of the GDF detector, it is shown that the proposed grouping algorithm, in fact, maximizes the AGSE lower bound for every group of users. Together with a fast computational method based on branch-and-bound, the theoretical analysis of the grouping algorithm enables the offline estimation of the computational cost and the performance of GDF detector. Simulation results on both small and large size problems are presented to verify the theoretical conclusions. All the results in this paper can be applied to the Decision Feedback (DF) detector by simply setting the maximum group size to 1.**

Keywords— **Multiuser detection, decision feedback, optimization methods, code division multiple access.**

I. Introduction

In synchronous Code Division Multiple Access (CDMA) communication systems, the near-far problem caused by the interuser interference has been widely studied. With the additive white Gaussian noise assumption and when the source signal is binary- or integer-valued, the conventional detector does not produce reliable decisions for the CDMA channel [?]. The computation of the optimal detection, however, is generally NP-hard and thus is exponential in the number of users [2], unless the signature wave form correlation matrix has a special structure [10] [9]. Several new algorithms have been proposed to provide reliable solutions with relatively low computational cost. Among the sub-optimal algorithm groups, the decision-driven detection methods, including decision feedback (DF) [5] [11], group detection [6], and multistage detection [3] [4], are popular. Although the DF method is simple and performs well, there are situations when a marginal increase in computation can provide significant improvement in performance [12].

The main drawback of DF is that detections are made for one user at a time; the decision on the strong user is obtained by treating the weak users as noise. However, when user chip sequences are correlated, this noise becomes biased, and thus is naturally harmful to the userwise detection. The idea of sequential group detection was first introduced by Varanasi in [6] and can be viewed as the Group Decision Feedback (GDF) detector. GDF detector divides users into several groups. The users with relatively high correlations are assigned to the same group, and the correlation between users in different groups are relatively low. Similar to DF detector, GDF detector makes decisions sequentially based on successive cancelation. However, instead of making decisions on single user at a time, GDF detector makes decisions groupwise, i.e., the decisions on users in the same group (the correlated users) are made simultaneously. The computational expense for a GDF detector is approximately exponential in the largest group size, and this is expected to be small if the largest group size is small.

In [6], the sizes of the groups are design parameters. However, in practice, given a user signal set, it is not easy for one to find the correlated users and assign them to groups. Since the largest group size is closely related to the overall computational cost, in this paper, we consider the largest group size as the only design parameter. A grouping and ordering algorithm is proposed to find the optimal size and users for each group. Theoretical results are given to show the optimality in terms of the Asymptotic Symmetric Energy (ASE). Together with a fast computational method modified from [12], the proposed GDF detection method provides an efficient way to improve the DF detection with marginal increase in computational cost. Simulation results on small and large size problems are presented to verify the theoretical conclusions.

The rest of the paper is organized as follows. In section II, we review the problem model and the theoretical results on the performance measure given in [6]. In section III, given the largest group size, a grouping and ordering algorithm is proposed to maximize the ASE of the system. Proof of optimality is given in the appendix. A fast computational method is proposed for the GDF detector and a theoretical upper bound on computational cost is derived. Simulation results on a small example as well as on a system of 100 users are presented in section IV. Conclusions are provided in section V.

II. Problem Formulation and Performance Measure of GDF Detector

A discrete-time equivalent model for the matched-filter outputs at the receiver of a CDMA channel is given by the

The Authors are with the Electrical and Computer Engineering Department, University of Connecticut, Storrs, CT06269, USA. E-mail:krishna@engr.uconn.edu

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K-length vector [2]

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{b} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{b} \in \{-1, +1\}^K$ denotes the vector of bits transmitted by the K active users. Here $\boldsymbol{H}$ is a nonnegative definite signature waveform correlation matrix, $\boldsymbol{n}$ is a real-valued zero-mean Gaussian random vector with a covariance matrix $\sigma^2\boldsymbol{H}$. It has been shown that this model holds for both baseband [2] and passband [11] channels with additive Gaussian noise.

When all the user signals are equally probable, the optimal solution of (1) is the output of a Maximum Likelihood (ML) detector [2]

$$\phi_{ML} : \hat{\boldsymbol{b}} = \arg \min_{\boldsymbol{b} \in \{-1,+1\}^K} \left(\boldsymbol{b}^T\boldsymbol{H}\boldsymbol{b} - 2\boldsymbol{y}^T\boldsymbol{b}\right) \tag{2}$$

which is generally NP-hard and exponentially complex to implement.

The sequential group detection based on the idea of successive cancellation was first introduced by Varanasi in [6]. Suppose users are partitioned into an ordered set of P groups, $G_0, \ldots, G_{P-1}$. The number of users in group G_i is denoted by $|G_i|$, and naturally $\sum_{i=0}^{P-1} |G_i| = K$. The decision on group $\{G_0\}$ is made by

$$\hat{\boldsymbol{b}}_{G_0} = \arg \min_{\boldsymbol{b}_{G_0} \in \{-1,+1\}^{|G_0|}} \left[\min_{\boldsymbol{b}_{\bar{G}_0}} \left(\boldsymbol{b}^T\boldsymbol{H}\boldsymbol{b} - 2\boldsymbol{y}^T\boldsymbol{b}\right) \right] \tag{3}$$

where $\boldsymbol{b}_{G_0}$ denotes the part of vector $\boldsymbol{b}$ that corresponds to users in group G_0, and $\bar{G}_0$ denotes the complement of G_0, i.e., the union of $G_1, \ldots, G_{P-1}$. The decisions of (3) are then used to subtract the multiple-access interference due to users in G_0 from the remaining decision statistics $\boldsymbol{y}_{\bar{G}_0}$. The detector for the next group G_1 is designed under the assumption that the multiple-access interference cancelation is perfect. This process of interference cancelation and group detection is carried out sequentially for users in groups $G_2, \ldots, G_{P-1}$, with the group detector for group G_i taking advantage of the decisions made by group detectors for $G_0, \ldots, G_{i-1}$. Denote the channel model for the user expurgated channel that only has users in groups $G_i, \ldots, G_{P-1}$ by

$$\boldsymbol{y}^{(i)} = \boldsymbol{H}^{(i)}\boldsymbol{b}^{(i)} + \boldsymbol{n}^{(i)} \tag{4}$$

The decisions on group G_i can be represented as

$$\hat{\boldsymbol{b}}_{G_i} = \arg \min_{\boldsymbol{b}^{(i)}_{G_i} \in \{-1,+1\}^{|G_i|}} \left[\min_{\boldsymbol{b}^{(i)}_{\bar{G}_i}} \left({\boldsymbol{b}^{(i)}}^T \boldsymbol{H}^{(i)} \boldsymbol{b}^{(i)} - 2{\boldsymbol{y}^{(i)}}^T \boldsymbol{b}^{(i)} \right) \right] \tag{5}$$

In multi-user detection, the Asymptotic Symmetric Energy (ASE) is an important performance measure. Define the probability that not all users are detected correctly as $P(\sigma, \phi)$, then the ASE for the detector ϕ [11] is given by

$$\eta(\phi) = sup \left\{ e \geq 0; \lim_{\sigma \to 0} \frac{P(\sigma, \phi)}{Q\left(\frac{\sqrt{e}}{\sigma}\right)} < \infty \right\} \tag{6}$$

where σ^2 is the additive noise variance (see (1)), and $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. The ASE for the optimal detector ϕ_{ML} is given by

$$\eta(\phi_{ML}) = d^2_{min} = \min_{\boldsymbol{e} \in \{-1,0,1\}^K \setminus \{0\}^K} \boldsymbol{e}^T\boldsymbol{H}\boldsymbol{e} \tag{7}$$

where d_{min} is known as the minimum distance of matrix $\boldsymbol{H}$ [8] and "\" is the set subtraction.

Similarly, we can define the Asymptotic Group Symmetric Energy (AGSE) for each user group. For a group detector, define the probability that not all users in group $\{G_i\}$ are detected correctly as $P_{G_i}(\sigma, \phi)$, and correspondingly we have

$$\eta_{G_i}(\phi) = sup \left\{ e \geq 0; \lim_{\sigma \to 0} \frac{P_{G_i}(\sigma, \phi)}{Q\left(\frac{\sqrt{e}}{\sigma}\right)} < \infty \right\} \tag{8}$$

as the AGSE for group $\{G_i\}$. Although an exact performance analysis of GDF detector is intractable [6], one can obtain upper and lower bounds for the AGSE of all groups. In the above description of the GDF detector, define $\boldsymbol{J}^{(i)} = \left[{\boldsymbol{H}^{(i)}}^{-1}\right]$, and denote $\boldsymbol{J}^{(i)}_{G_iG_i}$ to be the sub-matrix of $\boldsymbol{J}^{(i)}$ that only contains the columns and rows corresponding to users in G_i. Define $d_{G_i,min}$ to be the minimum distance of matrix $\left(\boldsymbol{J}^{(i)}_{G_iG_i}\right)^{-1}$, i.e.,

$$d^2_{G_i,min} = \min_{\boldsymbol{e} \in \{-1,0,1\}^{|G_i|} \setminus \{0\}^{|G_i|}} \boldsymbol{e}^T \left(\boldsymbol{J}^{(i)}_{G_iG_i}\right)^{-1} \boldsymbol{e} \tag{9}$$

Then the AGSE for group G_i can be bounded by

$$\min(d^2_{G_0,min}, \ldots, d^2_{G_i,min}) \leq \eta_{G_i}(\phi) \leq d^2_{G_i,min} \tag{10}$$

A similar result can be found in [6]. The upper bound in (10) is reached when all decisions on the users in group G_1 through group G_{i-1} are correct.

III. Optimal Grouping and Detection Order for GDF Detector

It is known that the performance of the decision-driven multi-user detector is significantly affected by the order of the users [?]. Since the overall computation for GDF detector is exponential in the maximum group size, which is defined by $|G|_{max} = \max(|G_0|, \ldots, |G_{P-1}|)$, in this section, we develop a grouping and ordering algorithm that maximizes the ASE of the GDF detector given $|G|_{max}$ as a design parameter.

Denote the Cholesky decomposition of $\boldsymbol{H}$ by $\boldsymbol{L}^T\boldsymbol{L} = \boldsymbol{H}$, where $\boldsymbol{L}$ is a lower triangular matrix. Multiply both sides of (1) by $(\boldsymbol{L}^{-1})^T$ to obtain the white noise model [5]

$$(\boldsymbol{L}^{-1})^T\boldsymbol{y} = \boldsymbol{L}\boldsymbol{b} + (\boldsymbol{L}^{-1})^T\boldsymbol{n} \tag{11}$$

Define $\tilde{\boldsymbol{y}} = (\boldsymbol{L}^{-1})^T\boldsymbol{y}$, $\tilde{\boldsymbol{n}} = (\boldsymbol{L}^{-1})^T\boldsymbol{n}$, partition the matrices and the vectors according to G_0 and $\bar{G}_0$ to obtain

$$\begin{bmatrix} \tilde{\boldsymbol{y}}_{G_0} \\ \tilde{\boldsymbol{y}}_{\bar{G}_0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{G_0G_0} & 0 \\ \boldsymbol{L}_{\bar{G}_0G_0} & \boldsymbol{L}_{\bar{G}_0\bar{G}_0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_{G_0} \\ \boldsymbol{b}_{\bar{G}_0} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{n}}_{G_0} \\ \tilde{\boldsymbol{n}}_{\bar{G}_0} \end{bmatrix} \tag{12}$$

Since $\boldsymbol{L}_{\bar{G}_0\bar{G}_0}$ is a full rank matrix by assumption, the decision on group G_0 in (3) can be written as

$$\hat{\boldsymbol{b}}_{G_0} = \arg \min_{\boldsymbol{b}_{G_0} \in \{-1,+1\}^{|G_0|}} \left\| \boldsymbol{L}_{G_0G_0}\boldsymbol{b}_{G_0} - \tilde{\boldsymbol{y}}_{G_0} \right\|_2^2 \quad (13)$$

Therefore, the AGSE of group G_0 is determined by the minimum distance of matrix $\boldsymbol{L}_{G_0G_0}^T \boldsymbol{L}_{G_0G_0}$. Since $\boldsymbol{H} = \boldsymbol{L}^T\boldsymbol{L}$, we have

$$\boldsymbol{L}_{G_0G_0}^T \boldsymbol{L}_{G_0G_0} = \left[(\boldsymbol{H}^{-1})_{G_0G_0}\right]^{-1} = \left[\boldsymbol{J}_{G_0G_0}^{(0)}\right]^{-1}$$
$$\eta_{G_0}(\phi_{GDFD}) = d_{G_0,min}^2 \quad (14)$$

A similar result can be obtained for group G_i. In the description of GDFD in section II, if we let $\boldsymbol{H}^{(i)} = \boldsymbol{L}^{(i)T}\boldsymbol{L}^{(i)}$, then $\boldsymbol{L}^{(i)T}_{G_i}\boldsymbol{L}^{(i)}_{G_i} = \left(\boldsymbol{J}_{G_iG_i}^{(i)}\right)^{-1}$. Since $\boldsymbol{H}^{(i)}$ is the south-east sub-diagonal matrix of $\boldsymbol{H}$, $\boldsymbol{L}^{(i)}$ is the south-east sub-diagonal matrix of $\boldsymbol{L}$ and $\boldsymbol{L}^{(i)}_{G_i} = \boldsymbol{L}_{G_i}$. Hence,

$$\boldsymbol{L}_{G_iG_i}^T \boldsymbol{L}_{G_iG_i} = \left(\boldsymbol{J}_{G_iG_i}^{(i)}\right)^{-1} \quad (15)$$

The above result shows that $d_{G_i,min}$ is determined by the diagonal block-matrix $\boldsymbol{L}_{G_i}$ of $\boldsymbol{L}$. Now, given all the decisions on group G_0 to group G_{i-1} are correct, denote the probability that not all the users in group G_i are detected correctly by $P_e(G_i|G_0,...,G_{i-1}) \approx Q\left(\frac{d_{G_i,min}}{\sigma}\right)$. The probability that not all the K users are detected correctly can be represented as

$$P(\sigma,\phi) \approx 1 - \prod_{i=0}^{P-1}\left[1 - Q\left(\frac{d_{G_i,min}}{\sigma}\right)\right] \quad (16)$$

Therefore, the ASE of GDF detector is given by

$$\eta(\phi_{GDFD}) = \min(d_{G_0,min}^2, ..., d_{G_{P-1},min}^2) \quad (17)$$

Since $|G|_{max}$ is given as a design parameter, the problem is then to find an optimal partition and detection order that maximizes $\min(d_{G_0,min}^2, ..., d_{G_{P-1},min}^2)$. Notice that different GDF detectors may have the same $|G|_{max}$ but different numbers of groups since P is not a design parameter.

Grouping and Order Algorithm : Find the optimal grouping and detection order via the following steps.

Step 1: Partition the K users into two groups $\{G_0\}$ and $\{\bar{G}_0\}$ with $|G_0| \leq |G|_{max}$. Among these partitions ($\{G_0\}$ and $|G_0|$ are not fixed), select the one that maximizes $d_{G_0,min}$ (which is the minimum distance of matrix $\left[\boldsymbol{J}_{G_0G_0}^{(0)}\right]^{-1}$).

Step 2: Partition the remaining $K - |G_0|$ users into two groups G_1 and $\bar{G}_1$ with $|G_1| \leq |G|_{max}$. Among these partitions, select the one that maximizes $d_{G_1,min}$.

Step 3: Continue this process until all the users are assigned to groups.

Example 1 : The algorithm is illustrated by the following 4-user example. Suppose the H matrix is given by

$$\boldsymbol{H} = \begin{bmatrix} 4.30 & 1.00 & 0.60 & 0.30 \\ 1.00 & 3.00 & 1.70 & 0.50 \\ 0.60 & 1.70 & 2.20 & 0.70 \\ 0.30 & 0.50 & 0.70 & 1.90 \end{bmatrix} \quad (18)$$

Assume that the desired maximum group size is $|G|_{max} = 2$. In step 1 of the algorithm, the possible choices for group G_0 and the resulting $d_{G_0,min}^2$ are shown in Table I. The

User(s)	0	1	2	3	0,1
$d_{G_0,min}^2$	3.96	1.62	1.14	1.67	1.69
User(s)	0,2	0,3	1,2	1,3	2,3
$d_{G_0,min}^2$	1.14	1.68	1.74	1.62	1.24

TABLE I
DIFFERENT CHOICES OF GROUP G_0 AND THE CORRESPONDING $d_{G_0,min}^2$

best choice for group G_0 is {user 0}. Then, for the user expurgated channel, we have

$$\boldsymbol{H}^{(1)} = \begin{bmatrix} 3.00 & 1.70 & 0.50 \\ 1.70 & 2.20 & 0.70 \\ 0.50 & 0.70 & 1.90 \end{bmatrix} \quad (19)$$

The possible choices for group G_1 and the resulting $d_{G_1,min}$ are shown in Table II. We can see that the best choice for

User(s)	1	2	3	1,2	1,3	2,3
$d_{G_1,min}^2$	1.69	1.14	1.68	1.78	1.68	1.24

TABLE II
DIFFERENT CHOICES OF GROUP G_1 AND THE CORRESPONDING $d_{G_1,min}$

group G_1 is {user 1, user 2}. Naturally {user 3} will be the last group. The resulting ASE for this partitioning and ordering is $\eta = 1.78$.

Note that the above example has 4 users and $|G|_{max} = 2$. One may think that partitioning users into 2 groups with 2 users in each group is a good choice. However, since user 0 is a strong user, it has to be detected first. And since user 1 and user 2 are seriously correlated, they have to be assigned to the same group. If, for example, we assign two groups as $\{user0, user3\}$ and $\{user1, user2\}$. As a punishment of detecting the weak user (user 3) first, we get $\eta = 1.68 < 1.78$.

Proposition 1 : The proposed grouping and ordering algorithm maximizes the ASE in (17).

See Appendix for the proof.

The proposed grouping and ordering algorithm is also optimal in the following sense.

Proposition 2 : The proposed grouping and ordering algorithm maximizes the performance lower bound in (10) for every group. In other words, suppose G is the grouping and ordering result obtained from the proposed algorithm, and G_k is one of the groups in G. Further suppose there is another group and detection sequence $\hat{G}$ with $\hat{G}_l$ being one of the groups in $\hat{G}$, and $\hat{G}_l = G_k$. Then the following result holds,

$$\min(d_{G_1,min}^2, \ldots, d_{G_k,min}^2) \geq \min(d_{\hat{G}_1,min}^2, \ldots, d_{\hat{G}_l,min}^2) \quad (20)$$

See Appendix for the proof.

In addition to the above 2 propositions, we can derive a fast computational method for GDF detector, which is a modified version of the method proposed in [12]. We propose the following steps for the group detection.

Computational Method for GDF Detector: Suppose the GDF detector has P groups, G_0, ..., G_{P-1}

1) Initialize $\tilde{\boldsymbol{y}}^{(1)} = (\boldsymbol{L}^{-1})^T \boldsymbol{y}$, $\boldsymbol{L}^{(1)} = \boldsymbol{L}$. Let $i = 1$;

2) Form the white noise system model for the userexpurgated channel, and partition the vectors and matrices according to group G_i and its complement $\bar{G}_i$ as

$$\begin{bmatrix} \tilde{\boldsymbol{y}}^{(i)}_{G_i} \\ \tilde{\boldsymbol{y}}^{(i)}_{\bar{G}_i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}^{(i)}_{G_i G_i} & 0 \\ \boldsymbol{L}^{(i)}_{\bar{G}_i G_i} & \boldsymbol{L}^{(i)}_{\bar{G}_i \bar{G}_i} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}^{(i)}_{G_i} \\ \boldsymbol{b}^{(i)}_{\bar{G}_i} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{n}}^{(i)}_{G_i} \\ \tilde{\boldsymbol{n}}^{(i)}_{\bar{G}_i} \end{bmatrix} \quad (21)$$

Find the decision on group G_i by

$$\hat{\boldsymbol{b}}_{G_i} = \arg \min_{\boldsymbol{b}_{G_i} \in \{-1,+1\}^{|G_i|}} \left\| \boldsymbol{L}_{G_i G_i} b_{G_i} - \tilde{\boldsymbol{y}}^{(i)}_{G_i} \right\|_2^2 \quad (22)$$

3) Compute $\tilde{\boldsymbol{y}}^{(i+1)}$ by

$$\tilde{\boldsymbol{y}}^{(i+1)} = \tilde{\boldsymbol{y}}^{(i)}_{\bar{G}_i} - \boldsymbol{L}^{(i)}_{\bar{G}_i G_i} \hat{\boldsymbol{b}}_{G_i} \quad (23)$$

Let

$$\boldsymbol{L}^{(i+1)} = \boldsymbol{L}^{(i)}_{\bar{G}_i \bar{G}_i} \quad (24)$$

4) Let $i = i+1$. If $i < P$, go to step 2; otherwise, stop the computation.

The computational cost for step 1 is $\frac{K(K+1)}{2}$ multiplications and $\frac{K(K-1)}{2}$ additions. Assume the computational cost for step 2 can be bounded by

$$\text{“}\times\text{”} \le M(|G_i|) \quad , \quad \text{“}+\text{”} \le S(|G_i|) \quad (25)$$

where “$\times$” denotes the number of multiplications and “$+$” denotes the number of additions. In step 3, since $\boldsymbol{b}$ can only take known discrete values, $\boldsymbol{Lb}$ can be precomputed and stored. Thus, only $|G_i| \sum_{k=i+1}^{P-1} |G_k|$ additions are needed. Therefore, the overall computational cost is bounded by

$$\begin{aligned} \text{“}\times\text{”} &\le \frac{K(K+1)}{2} + \sum_{k=0}^{P-1} \left[M(|G_k|) \right] \\ \text{“}+\text{”} &\le \frac{K(K-1)}{2} + \sum_{k=0}^{P-1} \left[S(|G_k|) + |G_k| \sum_{j=k+1}^{P-1} |G_j| \right] \end{aligned} \quad (26)$$

IV. Simulation Results

Example 1 - continued : In the previous 4-user example, $\eta(\phi_{GDFD}) = 1.78$. The ASE for optimal DDFD and the ML detector can be obtained from [11] as $\eta(\phi_{DDFD}) = 1.69$ and $\eta(\phi_{ML}) = 1.8$. The simulation results are shown in Figure 1, which are consistent with the theoretical analysis.

Example 2 : Suppose we have 100 users. The signature sequences for each user are binary and of length 115. They are generated randomly. The maximum group size

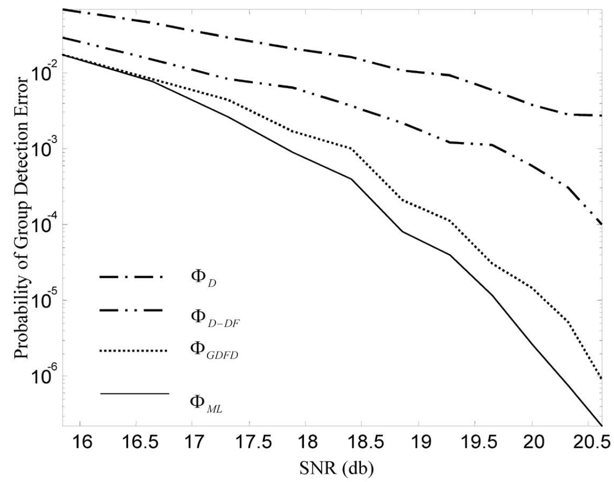

Fig. 1. Performance of various methods (4 users, 10000 Monte-Carlo runs. ϕ_D is the conventional decorrelator; ϕ_{D-DF} is the decorrelation-based decision feedback detector; ϕ_{GDFD} is the group decision feedback detector with $|G|_{max} = 2$; and ϕ_{ML} is the maximum likelihood detector.)

is assumed to be 3. Figure 2 shows one of the simulation results. The respective computational costs for the three detectors are

$$\begin{array}{lll} \phi_D & \text{“}\times\text{”} = 10000 & \text{“}+\text{”} = 9900 \\ \phi_{D-DFD} & \text{“}\times\text{”} = 5050 & \text{“}+\text{”} = 9900 \\ \phi_{GDFD} & \text{“}\times\text{”} = 5320 & \text{“}+\text{”} = 10020 \end{array} \quad (27)$$

Benifiting from the optimal grouping and the branch-and-bound-based computational method, GDFD shows a significant impreovement on the performance while the computational cost is even less than that of the conventional decorrelator. Due to the NP-hard nature of the optimal ML detector, the results on optimal detector could not be computed.

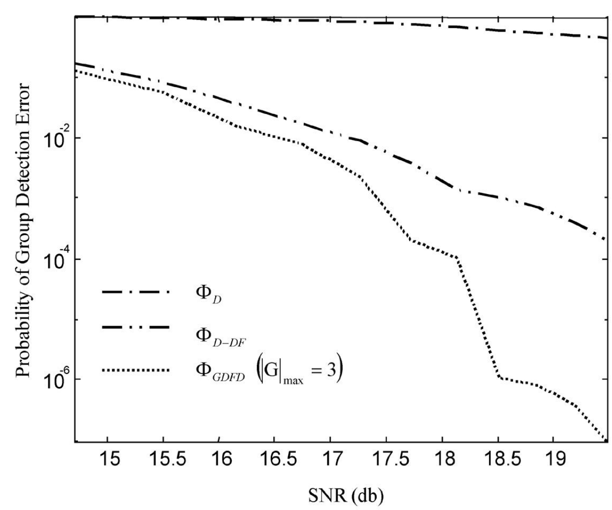

Fig. 2. Performance of various methods (100 users, 10000 Monte-Carlo runs. ϕ_D is the conventional decorrelator; ϕ_{D-DF} is the decorrelation-based decision feedback detector; ϕ_{GDFD} is the group decision feedback detector with $|G|_{max} = 3$)

V. Conclusion

An optimal grouping and ordering algorithm for Group Decision Feedback Detector is proposed. Together with a fast computational method based on the idea of branch and bound, the proposed algorithm provides a systematic way of improving the Decision Feedback Detector, especially when correlation exists among the users. Simulation results show that GDF detector with the optimal grouping and ordering algorithm provides a significant improvement over DF detector, while the increase in computational cost is marginal and even negative in some cases. The proposed method can be easily extended to finite-alphabet signals instead of binary ones.

Appendix

I. Pre-proved Lemmas

Before proving the propositions in this paper, we present the following three lemmas that will be used in the proof.

Lemma 1: Suppose $\boldsymbol{H} = \boldsymbol{L}^T\boldsymbol{L}$ is partitioned on two arbitrary diagonal elements as

$$\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{21}^T \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \tag{28}$$

For any permutation matrix $\boldsymbol{P}$ of the same size as $\boldsymbol{H}_{22}$, if

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{21}^T \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{P} \end{bmatrix} \\ = &\begin{bmatrix} \tilde{\boldsymbol{L}}_{11} & 0 \\ \tilde{\boldsymbol{L}}_{21} & \tilde{\boldsymbol{L}}_{22} \end{bmatrix}^T \begin{bmatrix} \tilde{\boldsymbol{L}}_{11} & 0 \\ \tilde{\boldsymbol{L}}_{21} & \tilde{\boldsymbol{L}}_{22} \end{bmatrix} \end{aligned} \tag{29}$$

then the following results hold.

$$\tilde{\boldsymbol{L}}_{11} = \boldsymbol{L}_{11} \quad , \quad \tilde{\boldsymbol{L}}_{22}^T\tilde{\boldsymbol{L}}_{22} = \boldsymbol{P}\boldsymbol{L}_{22}^T\boldsymbol{L}_{22}\boldsymbol{P} \tag{30}$$

The proof is quite straight forward and is therefore ignored in this paper. □

Lemma 2: Suppose $\boldsymbol{H}$ is a $m \times m$ symmetric and positive definite matrix. Suppose $\boldsymbol{H} = \boldsymbol{L}^T\boldsymbol{L}$ is the Cholesky decomposition. Partition $\boldsymbol{H}$ and $\boldsymbol{L}$ on the last (south-east) diagonal component as

$$\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{h}_{21}^T \\ \boldsymbol{h}_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ \boldsymbol{l}_{21} & l_{22} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ \boldsymbol{l}_{21} & l_{22} \end{bmatrix} \tag{31}$$

Now "move up" the last "user" to the first, denote the action and the new Cholesky decomposition matrix by

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ \boldsymbol{I} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{h}_{21}^T \\ \boldsymbol{h}_{21} & h_{22} \end{bmatrix} \begin{bmatrix} 0 & \boldsymbol{I} \\ 1 & 0 \end{bmatrix} \\ = &\begin{bmatrix} \tilde{l}_{11} & 0 \\ \tilde{\boldsymbol{l}}_{21} & \tilde{\boldsymbol{L}}_{22} \end{bmatrix}^T \begin{bmatrix} \tilde{l}_{11} & 0 \\ \tilde{\boldsymbol{l}}_{21} & \tilde{\boldsymbol{L}}_{22} \end{bmatrix} \end{aligned} \tag{32}$$

Then matrix $\tilde{\boldsymbol{L}}_{22}^T\tilde{\boldsymbol{L}}_{22} - \boldsymbol{L}_{11}^T\boldsymbol{L}_{11}$ is non-negative definite.

Proof : Substituting (31) into (32) yields

$$\tilde{\boldsymbol{L}}_{22}^T\tilde{\boldsymbol{L}}_{22} - \boldsymbol{L}_{11}^T\boldsymbol{L}_{11} = \boldsymbol{l}_{21}^T\boldsymbol{l}_{21} \geq 0 \tag{33}$$

□

Lemma 3: Suppose $\boldsymbol{L}$ and $\tilde{\boldsymbol{L}}$ are two lower triangular matrices of size $m \times m$, assume that $\boldsymbol{L}^T\boldsymbol{L} - \tilde{\boldsymbol{L}}^T\tilde{\boldsymbol{L}} \geq 0$. Partition $\boldsymbol{L}$ on an arbitrary diagonal component, and partition $\tilde{\boldsymbol{L}}$ accordingly as

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_{11} & 0 \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \quad , \quad \tilde{\boldsymbol{L}} = \begin{bmatrix} \tilde{\boldsymbol{L}}_{11} & 0 \\ \tilde{\boldsymbol{L}}_{21} & \tilde{\boldsymbol{L}}_{22} \end{bmatrix} \tag{34}$$

We have

$$\boldsymbol{L}_{11}^T\boldsymbol{L}_{11} - \tilde{\boldsymbol{L}}_{11}^T\tilde{\boldsymbol{L}}_{11} \geq 0 \quad , \quad \boldsymbol{L}_{22}^T\boldsymbol{L}_{22} - \tilde{\boldsymbol{L}}_{22}^T\tilde{\boldsymbol{L}}_{22} \geq 0 \tag{35}$$

Proof : Since $\boldsymbol{L}^T\boldsymbol{L} - \tilde{\boldsymbol{L}}^T\tilde{\boldsymbol{L}} \geq 0$, we can find a lower triangular matrix $\boldsymbol{C}$ which satisfies

$$\boldsymbol{L}^T\boldsymbol{L} = \tilde{\boldsymbol{L}}^T\left(\boldsymbol{I} + \boldsymbol{C}^T\boldsymbol{C}\right)\tilde{\boldsymbol{L}} \tag{36}$$

According to (34), partition $\boldsymbol{C}$ as

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_{11} & 0 \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{bmatrix} \tag{37}$$

Substitute (34)(37) into (36) to obtain

$$\begin{aligned} \boldsymbol{L}_{22}^T\boldsymbol{L}_{22} &= \tilde{\boldsymbol{L}}_{22}^T\left(\boldsymbol{I} + \boldsymbol{C}_{22}^T\boldsymbol{C}_{22}\right)\tilde{\boldsymbol{L}}_{22} \\ \boldsymbol{L}_{11}^T\boldsymbol{L}_{11} &= \tilde{\boldsymbol{L}}_{11}^T\left(\boldsymbol{I} + \boldsymbol{C}_{11}^T\boldsymbol{C}_{11}\right)\tilde{\boldsymbol{L}}_{11} + \triangle \end{aligned} \tag{38}$$

where $\triangle$ is a symmetric non-negative definite matrix. The proof is complete. □

Note that in Lemma 3, we can continue partitioning the sub-diagonal block matrices, and apply Lemma 3 iteratively to get a result similar to (35) for an arbitrary partition.

II. Proof of Proposition 1

Denote the optimal group and detection sequence determined by the proposed algorithm as G, which has groups $G_0, \ldots, G_{P-1}$. Denote the group decision feedback detector using detection sequence G by ϕ_{G-GDFD}. The idea of the proof can be summarized as follows. Suppose there is another group and detection sequence $G^{(i)}$, which has groups $G_0^{(i)}, \ldots, G_{P^{(i)}-1}^{(i)}$. Without loss of generality, assume $\forall j\ (0 \leq j < i)\ G_j^{(i)} = G_j$ (The superscript (i) means that the first i groups in $G^{(i)}$ are identical to the first i groups in G).

Now construct a new group and detection sequence $G^{(i+1)}$. The groups of $G^{(i+1)}$ are defined by

$$\begin{cases} G_j^{(i+1)} = G_j^{(i)} = G_j & 0 \leq j < i \\ G_j^{(i+1)} = G_j & j = i \\ G_j^{(i+1)} = G_{j-1}^{(i)} \setminus G_i & j > i \end{cases} \tag{39}$$

To simplify the notation, in the above construction, if $G_j^{(i+1)} = NULL$, we still keep group $G_j^{(i+1)}$ and define $d_{G_j^{(i+1)},min} = \infty$. Evidently, $G^{(i+1)}$ has one more group than $G^{(i)}$. The following result holds for $G^{(i+1)}$.

Proposition 3: If $G^{(i+1)}$ is constructed according to the above definition, then
(1) $\forall j \; (0 \le j < i)$, $d^2_{G_j^{(i+1)},min} = d^2_{G_j^{(i)},min}$.
(2) $d^2_{G_i^{(i+1)},min} \ge d^2_{G_i^{(i)},min}$.
(3) $\forall j \; (i < j \le P^{(i)})$, $d^2_{G_j^{(i+1)},min} \ge d^2_{G_{j-1}^{(i)},min}$.

Proof :
(1) For any $j < i$, the decision for group $G_j^{(i)}$ is made by treating the signal corresponding to $G_{j+1}^{(i)}$, ..., $G_{P^{(i)}-1}^{(i)}$ as noise and minimizing the probability of error in ML sense. Therefore, any swapping of users within groups of index larger than j will not affect the performance of $G_j^{(i)}$. This result can be formally proved by using Lemma 1.
(2) Since $G_j^{(i)} = G_j^{(i+1)}$ $(\forall j < i)$, this result can be directly obtained from the definition of the optimal grouping and ordering algorithm.
(3) The proof for this part is relatively tricky. In fact, the construction of $G^{(i+1)}$ from $G^{(i)}$ can be divided into three stages. Define the users in group G_i as K_0, ..., $K_{|G_i|-1}$. For the convenience of discussion, we first consider user K_0.

Stage 1 Suppose, in $G^{(i)}$, user K_0 belongs to group $G_j^{(i)}$ $(j \ge i)$. Define the the action "take out user K_0 from group $G_j^{(i)}$", which converts $G^{(i)}$ to $G^{(S1)}$, as,

$$\begin{cases} G_k^{(S1)} = G_k^{(i)} & k < j \\ G_k^{(S1)} = \{userK_0\} & k = j \\ G_k^{(S1)} = G_j^{(i)} - \{userK_0\} & k = j+1 \\ G_k^{(S1)} = G_{k-1}^{(i)} & k > j+1 \end{cases} \tag{40}$$

Stage 2 Now in $G^{(S1)}$, we have $G_j^{(S1)} = \{userK_0\}$. Define the action "move up user K_0 to follow group $G_{i-1}^{(S1)}$", which converts $G^{(S1)}$ to $G^{(S2)}$, as follows,

$$\begin{cases} G_k^{(S2)} = G_k^{(S1)} & k < i \\ G_k^{(S2)} = \{userK_0\} & k = i \\ G_k^{(S2)} = G_{k-1}^{(S1)} & i < k \le j \\ G_k^{(S2)} = G_k^{(S1)} & k > j \end{cases} \tag{41}$$

Continue performing the above two stages on all users K_0, ..., $K_{|G_i|-1}$. Denote the resulting group and detection sequence as $G^{(S3)}$. Denote the number of groups in $G^{(S3)}$ by $P^{(S3)}$.

Stage 3 In $G^{(S3)}$, combine groups $\{K_{|G_i|-1}\}$, ..., $\{K_0\}$, which converts $G^{(S3)}$ to $G^{(i+1)}$, as,

$$\begin{cases} G_k^{(i+1)} = G_k^{(S3)} & k < i \\ G_k^{(i+1)} = \{userK_0, ..., K_{|G_i|-1}\} & k = i \\ G_k^{(i+1)} = G_{k-|G_i|+1}^{(S3)} & k > i \end{cases} \tag{42}$$

In the first stage, without loss of generality, suppose user K_0 is the first user in group $G_j^{(i)}$. The "take out" action does not change the order of the users, thus the Cholesky decomposition matrix $\boldsymbol{L}$ remains unchanged. This shows that $\boldsymbol{L}_{G_{j+1}^{(S1)}G_{j+1}^{(S1)}}$ is the south-east diagonal sub-block of $\boldsymbol{L}_{G_j^{(i)}G_j^{(i)}}$. Therefore,

$$d^2_{G_{j+1}^{(S1)},min} \ge d^2_{G_j^{(i)},min} \tag{43}$$

In the second stage, since the "minimum distance" of a sub-block is the performance measure for the corresponding user group given all the user groups with smaller indices are correctly detected, putting more users into the detected user list will result in a better performance and a larger "minimum distance". In fact, from Lemma 2 and Lemma 3, for any groups $G_k^{(S2)} = G_{k-1}^{(S1)}$, $i < k \le j$, we have,

$$\boldsymbol{L}^T_{G_k^{(S2)}G_k^{(S2)}}\boldsymbol{L}_{G_k^{(S2)}G_k^{(S2)}} - \boldsymbol{L}^T_{G_{k-1}^{(S1)}G_{k-1}^{(S1)}}\boldsymbol{L}_{G_{k-1}^{(S1)}G_{k-1}^{(S1)}} \ge 0 \tag{44}$$

Therefore,

$$d^2_{G_k^{(S2)},min} \ge d^2_{G_{k-1}^{(S1)},min} \tag{45}$$

Hence, in $G^{(i+1)}$, for any $j > i$, $d^2_{G_j^{(i+1)},min} \ge d^2_{G_{j-1}^{(i)},min}$, which proves part (3) of proposition 3.
□

Based on proposition 3, suppose $\eta(\phi_{G^{(i+1)}-GDFD}) = d^2_{G_j^{(i+1)},min}$. Then, we have

$$\eta(\phi_{G^{(i+1)}-GDFD}) \ge \eta(\phi_{G^{(i)}-GDFD}) \tag{46}$$

By iteratively using the above construction procedure in the proof of Proposition 1, we will finally get $G^{(P)} = G$ and

$$\eta(\phi_{G^{(P)}-GDFD}) \ge \eta(\phi_{G^{(i)}-GDFD}) \tag{47}$$

which completes the proof. □

III. Proof of Proposition 2

In the above proof for proposition 1, let $G^{(i)} = \hat{G}$. Construct $G^{(i+1)}$ using the same procedure. Note that $G_l^{(i)} = \hat{G}_l = G_k$, and $G_k \cap G_i = NULL$. Therefore, in $G^{(i+1)}$, we have $G_{l+1}^{(i+1)} = G_k$. Suppose $\min(d^2_{G_0^{(i+1)},min}, \ldots, d^2_{G_{l+1}^{(i+1)},min}) = d^2_{G_j^{(i+1)},min}$. From Proposition 3,

- If $j < i$, we have $d^2_{G_j^{(i+1)},min} = d^2_{G_j^{(i)},min} \ge \min(d^2_{\hat{G}_0,min}, \ldots, d^2_{\hat{G}_l,min})$.
- If $j = i$, we have $d^2_{G_i^{(i+1)},min} \ge d^2_{G_i^{(i)},min} \ge \min(d^2_{\hat{G}_0,min}, \ldots, d^2_{\hat{G}_l,min})$.
- If $j > i$, we have $d^2_{G_j^{(i+1)},min} \ge d^2_{G_{j-1}^{(i)},min} \ge \min(d^2_{\hat{G}_0,min}, \ldots, d^2_{\hat{G}_l,min})$.

Hence,

$$\min(d^2_{G_0^{(i+1)},min}, \ldots, d^2_{G_{l+1}^{(i+1)},min}) \ge \min(d^2_{\hat{G}_0,min}, \ldots, d^2_{\hat{G}_l,min}) \tag{48}$$

By iteratively using the construction procedure, we will finally get $G^{(P)} = G$ which satisfies

$$\min(d^2_{G_0,min}, \ldots, d^2_{G_k,min}) \ge \min(d^2_{\hat{G}_0,min}, \ldots, d^2_{\hat{G}_l,min}) \tag{49}$$

Hence the proof is complete. □

References

[1] S. Verdu, *Minimum Probability of Error for asynchronous Gaussian Multiple-Access Channel*, IEEE Trans. Inform. Theory, vol. IT-32, pp. 85–96, Jan. 1986.

[2] R. Lupas and S. Verdu, *Linear multiuser detectors for synchronous code-devision multiple-access channels*, IEEE Trans. Inform. Theory, vol. 35, pp. 123–136, Jan. 1989.

[3] M. K. Varanasi, *Multistage detection for asynchronous code-division multiple-access communications*, IEEE Trans. Comun., vol. 38, pp. 509–519, Apr. 1990.

[4] M. K. Varanasi and B. Aazhang, *Near-optimum detection in synchronous code-division multiple access systems*, IEEE Trans. Comun., vol. 39, pp. 725–736, May 1991.

[5] A. Duel-Hallen, *Decorrelating Decision-Feedback Multiuser Detector for Synchronous Code-Division Multiple-Access Channel*, IEEE Trans. Comun., vol. 41, pp. 285–290, Feb. 1993.

[6] M. K. Varanasi, *Group Detection for Synchronous Gaussian Code-Division Multiple-Access Channels*, IEEE Trans. Inform. Theory, vol. 41, pp. 1083–1096, July 1995.

[7] A. Duel-Hallen, *A Family of Multiuser Decision-Feedback Detectors for Asynchronous Code-Division Multiple-Access Channels*, IEEE Trans. Comun., vol. 43, pp. 421–432, Feb./Mar./April 1995.

[8] C. Schlegel and L. Wei, *A simple way to compute the minimum distance in multiuser CDMA systems*, IEEE Trans. Communications, vol. 45, pp. 532–535, May 1997.

[9] C. SanKaran and A. Ephremides, *Solving a Class of Optimum Multiuser Detection Problems with Polynomial Complexity*, IEEE Trans. Inform. Theory, vol. 44, pp. 1958-1961, Sep. 1998.

[10] S. Ulukus, R. Yates, *Optimum multiuser detection is tractable for synchronous CDMA systems using M-sequences*, IEEE Commun. Lett., vol. 2, pp. 89-91, 1998.

[11] M. K. Varanasi, *Decision feedback multiuser detection: a systematic approach*, IEEE Trans. Inform. Theory, vol. 45, pp. 219–240, Jan. 1999.

[12] J. Luo, K. Pattipati, P. Willett, G. Levchuk, *Fast Optimal and Sub-optimal Any-Time Algorithms for CWMA Multiuser Detection based on Branch and Bound*, submitted to IEEE Trans. Commun., July 2000.