JOINT BOX-CONSTRAINT AND DEREGULARIZATION IN MULTIUSER DETECTION

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ABSTRACT

Multiuser detection can be described as a quadratic optimization problem with binary constraint. Many techniques are available to find approximate solution to this problem. These techniques can be characterized in terms of complexity and detection performance. The "efficient frontier" of known techniques include the decision-feedback (DF), branch-and-bound (BB) and probabilistic data association (PDA) detectors. We propose a novel iterative multiuser detection technique based on joint deregularized and box-constrained solution to quadratic optimization with iterations similar to that used in the nonstationary Tikhonov iterated algorithm. The deregularization maximizes the energy of the solution; this is opposite to the Tikhonov regularization where the energy is minimized. However, combined with box-constraints, the deregularization forces the solution to be close to the binary set. Our development improves the "efficient frontier" in multiuser detection, which is illustrated by simulation results.

1. INTRODUCTION

In multiple-access CDMA systems, multiuser detection is capable of providing high detection performance [1]. The multiuser detection theory has been well developed in the past two decades. However, the complexity of multiuser detection is still a very important issue. The complexity issue has been addressed in recent papers [2, 3, 4] and others, where new efficient multiuser detectors were proposed. Comparison of advanced multiuser techniques in [5], in terms of group detection error and complexity has shown that an "efficient frontier" of multiuser detectors is primarily composed of the decision-feedback (DF) detector [6], probabilistic data association (PDA) detector [2], and branch and bound (BB) detector [3]. The DF detector is the simplest one, the BB detector provides the best detection performance, while the PDA detector gives a good detection performance with simpler implementation than the BB detector [5].

In this paper, we propose a novel algorithm that solves the quadratic optimization problem incident to multiuser detection with near optimal detection performance and low computational complexity. The novel iterative multiuser detection technique is based on joint deregularized and box-constrained solution to quadratic optimization with iterations similar to that used in the nonstationary Tikhonov iterated algorithm [7, 8]. The deregularization maximizes the energy of the solution; this is opposite to the Tikhonov regularization where the energy is minimized. However, combined with box-constraints, the deregularization forces the solution to be close to the binary set. The proposed algorithm exploits both the advantages of the bootstrap detector [9] based on the nonstationary iterated Tikhonov regularization, and the box-constrained algorithm based on Gauss-Seidel (GS) iterations [10]. It achieves a performance very close to the BB detector with a complexity lower than the PDA detector; this significantly improves the "efficient frontier" defined in [5].

Although only multiuser detection problems are addressed in this paper, the algorithms are applicable to a wide class of applications since many communication problems can be reduced to quadratic optimization with similar constraints.

2. PROBLEM FORMULATION

The matched-filter output at a symbol synchronous CDMA receiver is given by the K-length vector

$$\mathbf{y} = \mathbf{R}\mathbf{b} + \mathbf{n} \tag{1}$$

where the vector $\mathbf{b} \in \{-1,+1\}^K$ contains bits transmitted by the K users, $\mathbf{R}$ is a positive definite signature waveform correlation matrix, and $\mathbf{n}$ is a real-valued zero-mean Gaussian random vector with covariance matrix $\sigma^2\mathbf{R}$. The model (1) also describes multiuser communications in multi-antenna, flat and frequency-selective fading channels [11]. The MIMO communication [12] and other communication scenarios (see, e.g., [13]) can also be modeled by (1). For simplicity, we concentrate on multiuser detection with BPSK modulation in this paper.

The optimal ML multiuser detector estimates the vector $\mathbf{b}$ by minimizing the following quadratic function [1] with binary constraints, $\mathbf{b} \in \{-1,+1\}^K$,

$$\hat{\mathbf{b}} = \arg \min_{\mathbf{b}\in\{-1,+1\}^K} \left\{ \frac{1}{2}\mathbf{b}^T\mathbf{R}\mathbf{b} - \mathbf{y}^T\mathbf{b} \right\}. \tag{2}$$

Although the ML detector provides the best detection performance, it is not practical due to its high complexity [1]. The decorrelating detector is a relatively simple technique that solves the same problem with no constraint, $\mathbf{b} \in \mathbb{R}^K$ [1]. However, the performance of the decorrelating detector is not always satisfactory. In general, to achieve a better performance, one should tighten the constraint on the solution set, which consequently results in an increased complexity of the multiuser detector.

The performance and complexity trade-off can be measured by the "efficient frontier", defined in [5]. Among the multiuser detectors that touch the efficient frontier, the BB detector provides the optimal performance with a low average complexity. Unfortunately, its worst case complexity is exponential in the number of users, which makes it infeasible for practical implementation. On the other hand, the DF detector provides good performance with a low complexity, while the PDA detector achieves near optimal performance with a moderate complexity.

3. THE BOOTSTRAP AND THE BOX-CONSTRAINED DETECTORS

In this section, we introduce the recently proposed bootstrap detector [9] and the box-constrained detector based on GS iterations [10]. Although both the detectors are able to provide high detection performance, none of the individual detectors can touch the efficient frontier. However, we will show that the combination of the two methods complemented with deregularization gives an outstanding suboptimal detector.

3.1 Bootstrap Detector

The MMSE detector solves the following unconstrained quadratic optimization problem [1]

$$\hat{\mathbf{b}} = \arg \min_{\mathbf{b}\in\mathbb{R}^K} \left\{ \frac{1}{2}\mathbf{b}^T\mathbf{R}\mathbf{b} - \mathbf{y}^T\mathbf{b} + \frac{\lambda}{2}\mathbf{b}^T\mathbf{b} \right\} \tag{3}$$

with the regularization term $(\lambda/2)\mathbf{b}^T\mathbf{b}$ that promotes solutions with small energy; $\lambda = \sigma^2 > 0$ is the regularization parameter. A more accurate solution to the initial optimization problem is obtained by using the iterated Tikhonov regularization [7, 8]. In this method, (3) is generalized to a sequence of unconstrained optimization, with the nth iteration given by the following equation

$$\hat{\mathbf{b}}^{(n)} = \arg\min_{\mathbf{b}\in \mathbb{R}^K} \left\{ \frac{1}{2}\mathbf{b}^T\mathbf{R}\mathbf{b} - (\mathbf{y}+\lambda\tilde{\mathbf{b}}^{(n-1)})^T\mathbf{b} + \frac{\lambda}{2}\mathbf{b}^T\mathbf{b} \right\} \quad (4)$$

where $\hat{\mathbf{b}}^{(0)} = \mathbf{0}$, $\tilde{\mathbf{b}}^{(n-1)} = \hat{\mathbf{b}}^{(n-1)}$, and $\lambda > 0$. Compared with (3), equation (4) allows the solution to be directed toward the solution $\tilde{\mathbf{b}}^{(n-1)}$ found at the previous iteration. This becomes clearer when the solution of the problem in (4) is represented as [7]

$$\hat{\mathbf{b}}^{(n)} = \tilde{\mathbf{b}}^{(n-1)} + \Delta_{\mathbf{b}}^{(n-1)} \quad (5)$$

where $\Delta_{\mathbf{b}}^{(n-1)}$ is the solution of

$$(\mathbf{R}+\lambda\mathbf{I})\Delta_{\mathbf{b}}^{(n-1)} = \mathbf{r}^{(n-1)} \quad (6)$$

with $\mathbf{r}^{(n-1)}$ being the residual vector, $\mathbf{r}^{(n-1)} = \mathbf{y} - \mathbf{R}\tilde{\mathbf{b}}^{(n-1)}$.

It is seen that for $n = 1$ and $\lambda = \sigma^2$, we obtain the MMSE solution, while for $n > 1$, the new solution $\hat{\mathbf{b}}^{(n)}$ lies in the vicinity of the solution $\hat{\mathbf{b}}^{(n-1)}$ found in the previous iteration.

The bootstrap multiuser detector proposed in [9] is similar to the algorithm presented in (4), however, it differs from (4) on the following two aspects.

1) Regularization parameter λ varies with iterations, $\lambda_0 = \sigma^2$, $\lambda_n = n\sigma^2$ for $n \geq 1$. This is known as the nonstationary iterated Tikhonov regularization [8].

2) At every iteration, box-constraints are used to obtain a semi-hard version of $\hat{\mathbf{b}}^{(n)}$:

$$\tilde{b}_k^{(n)} = \begin{cases} +1 & \text{if } \hat{b}_k^{(n)} > n^{\alpha} \\ -1 & \text{if } \hat{b}_k^{(n)} < -n^{\alpha} \\ \hat{b}_k^{(n)} & \text{otherwise} \end{cases} \quad (7)$$

where $\hat{b}_k^{(n)}$ and $\tilde{b}_k^{(n)}$ are respectively the kth elements of vectors $\hat{\mathbf{b}}^{(n)}$ and $\tilde{\mathbf{b}}^{(n)}$, and $\alpha < 0$ is a pre-determined parameter of the method. The final solution is obtained via $\hat{\mathbf{b}} = \text{sign}\{\tilde{\mathbf{b}}^{(N)}\}$ where N is the number of iterations.

The solution to (4) at the nth iteration ($n > 0$) can be regarded as an unconstrained solution to the following equation

$$(\mathbf{R}+\lambda\mathbf{I})\,\hat{\mathbf{b}}^{(n)} = \mathbf{y} + \lambda\tilde{\mathbf{b}}^{(n-1)}. \quad (8)$$

Obtaining a direct solution of (8) requires K^3 FLOPS. Consequently, such a direct implementation of the bootstrap detector with N iterations requires $(N+1)K^3$ FLOPS.

In order to reduce the complexity of the bootstrap detector, (8) can be solved approximately using the Gauss-Seidel (GS) iterations [14] within a bootstrap iteration. Note that GS iterations are extensively used in signal processing, and have been an efficient technique in multiuser detection [15], MIMO detection [16], and adaptive filtering [17]. Let $\mathbf{R}\mathbf{b} = \mathbf{y}$ be the equation to be solved. Starting from an arbitrary initial vector, say $\mathbf{b}^{(0)} = \mathbf{0}$, the $(j+1)$th GS iteration is given by

$$b_i^{(j+1)} = \frac{1}{\mathbf{R}_{ii}}\left(y_i - \sum_{k=1}^{i-1}\mathbf{R}_{ik}b_k^{(j+1)} - \sum_{k=i+1}^{K}\mathbf{R}_{ik}b_k^{(j)} \right) \quad (9)$$

where y_i is i-th element of the vector $\mathbf{y}$, $b_i^{(j)}$ is i-th element of the vector $\mathbf{b}^{(j)}$, $\mathbf{R}_{ik}$ is (i,k)-th element of the matrix $\mathbf{R}$, and $i = 0,\dots,K-1$. Assume that N_{GS} Gauss-Seidel iterations are used for every bootstrap iteration, $j = 1,\dots,N_{GS}$, then the complexity of the bootstrap detector with N bootstrap iterations is in the order of $2(N+1)N_{GS}K^2$ FLOPS. If we choose N_{GS} to be less than $K/2$, the complexity of the detector with GS iterations will be less than that of the direct implementation.

To simplify the algorithm further, when solving equation (8) for the $(n+1)$th bootstrap iteration, we initialize the GS iteration with the semi-hard solution (7) obtained from the nth bootstrap iteration. When the initial value is close to the equilibrium, the number of GS iterations for solving (8) can be significantly reduced. However, to ensure an accurate solution at the initial part of the bootstrap detector, we use $N_{GS}^{(init)}$ GS iterations for the first step, which solves the classical MMSE detection problem, and use $N_{GS}^{(bootstrap)}$ GS iterations for other bootstrap updates. The overall complexity of the algorithm is $2\left(N_{GS}^{(init)} + NN_{GS}^{(bootstrap)}\right)K^2$ FLOPS.

Below we will compare the performance of the three detectors: 1) a bootstrap detector with direct implementation (direct iterated); 2) a bootstrap detector implemented with GS iterations with zero initialization of the solution at every bootstrap iteration (GS-zero iterated); and 3) a bootstrap detector implemented with GS iterations and the semi-hard initialization of the solution at every bootstrap iteration (GS-SH iterated).

3.2 Box-Constrained Detector with Nonlinear GS Iterations

A box-constrained multiuser detector solves the following optimization problem

$$\hat{\mathbf{b}} = \arg\min_{\mathbf{b}\in[-1,+1]^K} \left\{ \frac{1}{2}\mathbf{b}^T\mathbf{R}\mathbf{b} - \mathbf{y}^T\mathbf{b} \right\}. \quad (10)$$

In the box-constrained detector with nonlinear GS iterations [10], every such iteration consists of two steps. At the first step, the classic GS iteration is performed:

$$b_i^{(j+1)} = \frac{1}{\mathbf{R}_{ii}}\left(y_i - \sum_{k=1}^{i-1}\mathbf{R}_{ik}\tilde{b}_k^{(j+1)} - \sum_{k=i+1}^{K}\mathbf{R}_{ik}\tilde{b}_k^{(j)} \right). \quad (11)$$

At the second step, the semi-hard update is performed:

$$\tilde{b}_i^{(j+1)} = \begin{cases} +1 & \text{if } b_i^{(j+1)} > +1 \\ -1 & \text{if } b_i^{(j+1)} < -1 \\ b_i^{(j+1)} & \text{otherwise.} \end{cases} \quad (12)$$

4. DEREGULARIZATION AND PROPOSED DETECTOR

The box-constraint tightens the solution set which results in significant improvement of the detection performance over unconstrained decorrelating and MMSE detectors. An additional tightening of the solution set could further improve the performance. Note that the constraint on each $\mathbf{b} \in \{-1,+1\}^K$ is equivalent to $b_i^2 = 1$ which implies $\mathbf{b}^T\mathbf{b} = K$ [10]. Then, the additional tightening of the solution set can be achieved by introducing the following optimization problem

$$\hat{\mathbf{b}} = \arg\min_{\mathbf{b}\in[-1,+1]^K} \left\{ \frac{1}{2}\mathbf{b}^T\mathbf{R}\mathbf{b} - \mathbf{y}^T\mathbf{b} + \frac{\lambda}{2}(K - \mathbf{b}^T\mathbf{b}) \right\} \quad (13)$$

where $\lambda > 0$. Solving (13) is equivalent to solving the problem

$$\hat{\mathbf{b}} = \arg\min_{\mathbf{b}\in[-1,+1]^K} \left\{ \frac{1}{2}\mathbf{b}^T(\mathbf{R}-\lambda\mathbf{I})\mathbf{b} - \mathbf{y}^T\mathbf{b} \right\}. \quad (14)$$

The replacement of the matrix $\mathbf{R}$ by the matrix $\mathbf{R}-\lambda\mathbf{I}$ implies that we solve a deregularized optimization problem. Note that

$(K - \mathbf{b}^T\mathbf{b}) \geq 0$ when the box-constraint is applied, i.e., for any $\mathbf{b} \in [-1,+1]^K$. Therefore, when the box-constraints are considered, classical regularization used in the MMSE detector does not always result in a performance improvement. Instead, the attempt of minimizing the objective function with an additional term $(\lambda/2)\mathbf{b}^T\mathbf{b}$ (as done in (3)) can make the detection performance even worse. It is intuitively clear that if a desired solution belongs to the binary set $\{-1,+1\}^K$ and we constraint solutions to be in the K-dimensional box $[-1,+1]^K$, then a solution with higher energy $\mathbf{b}^T\mathbf{b}$ is preferable as it lies closer to the binary set. Thus we are interested in an objective function that promotes solutions with higher energy. To achieve this, we propose the deregularization (14). The deregularization parameter λ is chosen as in the MMSE detector, $\lambda = \sigma^2$. This optimization can be implemented by using the box-constrained algorithm with nonlinear GS iterations (see Section 3.2) with the matrix $\mathbf{R}$ replaced by $\mathbf{R} - \lambda\mathbf{I}$. The final solution $\tilde{\mathbf{b}}$ can be obtained by projecting the vector $\hat{\mathbf{b}}$ to the binary constraint set, i.e., $\tilde{\mathbf{b}} = \text{sign}[\hat{\mathbf{b}}]$.

This approach results in a novel multiuser detector with improved performance with respect to the box-constrained multiuser detector. However, the detection performance can be further improved by applying iterations similar to those in the bootstrap detector. Specifically, we now apply the combination of the deregularization and box-constraint to bootstrap iterations.

At the zero iteration, the proposed detector solves the optimization problem (14) by using $N_{GS}^{(init)}$ nonlinear GS iterations described by (11) and (12). In further bootstrap iterations ($n = 1, \ldots, N$), the detector solves the optimization problem

$$\hat{\mathbf{b}}^{(n)} = \arg \min_{\mathbf{b} \in [-1,+1]^K} \left\{ \frac{1}{2}\mathbf{b}^T(\mathbf{R} - \lambda_n\mathbf{I})\mathbf{b} - (\mathbf{y} + \lambda_n\hat{\mathbf{b}}^{(n-1)})^T\mathbf{b} \right\} \quad (15)$$

which is implemented by the box-constrained algorithm with the matrix $\mathbf{R}$ replaced by $\mathbf{R} - \lambda_n\mathbf{I}$ and vector $\mathbf{y}$ replaced by $\mathbf{y} + \lambda\hat{\mathbf{b}}^{(n-1)}$; thus, the following equation is solved with box-constraint:

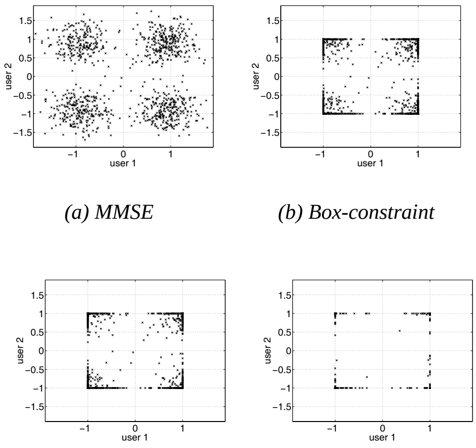
$$(\mathbf{R} - \lambda_n\mathbf{I})\,\hat{\mathbf{b}}^{(n)} = \mathbf{y} + \lambda_n\hat{\mathbf{b}}^{(n-1)}. \quad (16)$$

For solving this equation at every bootstrap iteration, we use $N_{GS}^{(bootstrap)}$ nonlinear GS iterations described by (11) and (12); typically, $N_{GS}^{(bootstrap)} << N_{GS}^{(init)}$. The deregularization parameter is $\lambda_n = n\sigma^2$. The final solution is obtained by applying the hard decision: $\tilde{\mathbf{b}} = \text{sign}[\hat{\mathbf{b}}^{(N)}]$.

5. NUMERICAL EXAMPLES

In this section, we present computer simulation examples to show the detection performance and complexity of the proposed multiuser detector in comparison with the BB, PDA, and DF detectors. In Example 1 below, we demonstrate constellation diagrams for different multiuser detectors, including the proposed GS-BD (Box-constrained with Gauss-Seidel iterations and non-stationary iterated deregularization) iterated detector, for a simple two-user scenario. In Examples 2,3, and 4, we use the simulation scenario from [5]. In this highly loaded scenario, there are $K = 60$ users with random binary spreading codes with the spreading factor $SF = 63$. The spreading codes are randomly generated, and then kept unchanged over all simulation trials.

Example 1. Fig.1 compares constellation diagrams in different multiuser detectors for a two-user scenario ($K = 2$) at a signal-to-noise ratio (SNR) of 7 dB. This illustrates the performance improvement due to introduction of box-constraints, the proposed deregularization, and bootstrap iterations. For the MMSE detector with the regularization (3), we can see an approximately symmetrical scattering of symbol estimates around the optimal set $\{-1,+1\}^2$. In the box-constrained detector, all symbol estimates are placed within the box $[-1,+1]^2$. In the box-constrained detector with deregularization (GS-BD, $N = 0$), the deregularization (14) forces the symbol estimates to have a high energy, consequently, they are on average closer to the optimal set than in the box-constrained detector. In the GS-BD iterated detector, after $N = 4$ bootstrap iterations, most of symbol estimates are on the boundary of the closure of the binary set, and close to the corner points.



(a) MMSE *(b) Box-constraint*

(c) GS-BD, $N = 0$ *(d) GS-BD iterated, $N = 4$*

Figure 1: Constellation diagrams for a two-user scenario; SNR $= 7$ dB.

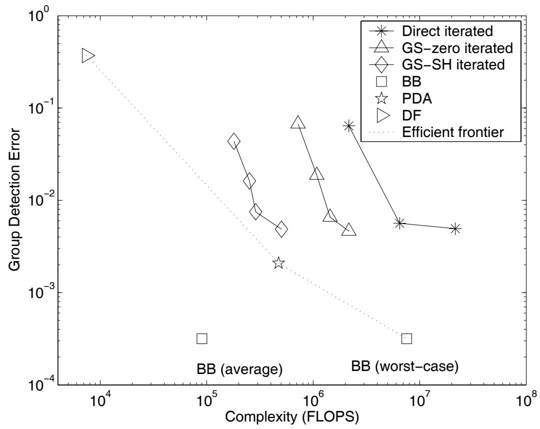


Figure 2: Group detection error vs worst-case complexity; SNR=10 dB, $K = 60$, SF=63.

Example 2. Fig.2 shows the group detection error relative to the detector complexity at SNR=10 dB. In the direct bootstrap detector, the number of bootstrap iterations takes on values $N = 10$, 20, and 30, the parameter $\alpha = -0.3$. As N increases, the complexity increases and the detection performance improves; however, further increase in N beyond $N = 30$ does not improve the detection performance. In the GS-zero bootstrap detector, the number of GS iterations is either $N_{GS} = 5$ or $N_{GS} = 10$, and the number of bootstrap iterations, $N = 10$ or $N = 30$. GS iterations reduce the complexity without degradation in the detection performance. The GS-SH bootstrap detector allows further reduction in the complexity without performance degradation. In this detector, the parameter $N_{GS}^{(init)} = 5, 10$ and $N_{GS}^{(bootstrap)} = 1, 2$ is used. It is interesting to observe that, for non-zero bootstrap iterations, usually only two GS iterations are enough to achieve the best performance. We also considered the initialization of the bootstrap detector by the vector sign$[\mathbf{y}]$ (matched-filter solutions); however, this did not improve the

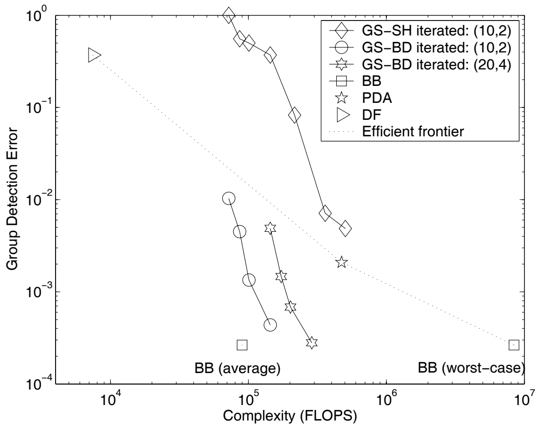


Figure 3: Group detection error vs worst-case complexity for the GS-BD iterated detector; SNR=10 dB, $K = 60$, SF=63.

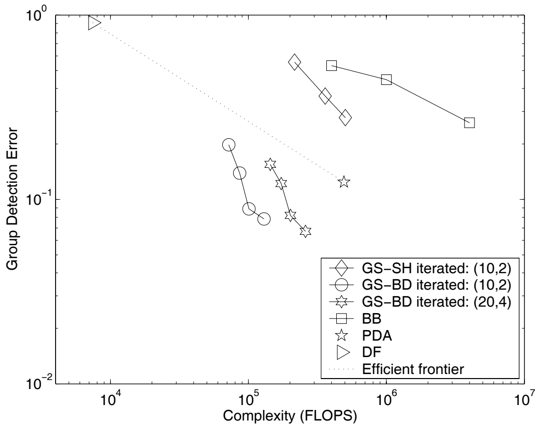


Figure 4: Group detection error vs worst-case complexity; SNR=7 dB, $K = 60$, SF=63.

detector performance. The DF detector has the lowest complexity, but its detection performance is poor. Although the BB detector gives the optimal performance with a low average complexity, its worst-case complexity is exponential in the number of users. It is seen that the GS-SH bootstrap detector outperforms the BB detector and the PDA detector in the worst-case complexity. However, the detection performance of the bootstrap detectors cannot achieve that of the BB and PDA detectors.

Example 3. Fig.3 shows the performance of the proposed GS-BD iterated detector. The number of bootstrap iterations in the detector varies as $N = 0, 1, 2, 4$. The other parameters are: $(N_{GS}^{(init)}, N_{GS}^{(bootstrap)}) = (10, 2)$ or $(20, 4)$. We can see that the GS-BD iterated detector significantly outperforms the GS-SH bootstrap detector (the most efficient version of the bootstrap detector) in both the detection performance and worst-case complexity. It also outperforms other detectors in the worst-case complexity. The only exception is the DF detector, but its detection performance is poor in this scenario. It is also seen that the use of box-constraints and deregularization allows reduction in the number of bootstrap iterations from 30 to 4 to achieve the best possible performance. With 4 bootstrap iterations, it achieves the same performance as the BB detector, i.e., it achieves the performance of the optimal ML detector. The complexity of the GS-BD iterated detector, approaching the optimal ML performance, is approximately $K^{3.2}$ FLOPS.

Example 4. Fig.4 demonstrates the performance at SNR=7 dB. At low SNRs, the complexity of the BB detector significantly increases. Therefore, simulation results are shown for a BB detector with upper bounded complexity [5] (and, as a result, a worse detection performance) for several complexity bounds. The detection performance of the proposed detector is significantly better than that of the other detectors. The complexity of the proposed detector is significantly less than that of other detectors, except the DF detector which, however, possesses a poor detection performance.

6. CONCLUSIONS

We have considered multiuser detection as the quadratic optimization problem and proposed its solution based on combined box-constrained minimization and non-stationary iterated deregularization. The latter is based on maximizing the solution energy, which is opposite to the classical MMSE detector with the Tikhonov regularization, where the energy of the solution is minimized. The deregularization when combined with box-constraints well suites to the formulation of the optimal ML detection. This combination of constraints and deregularization forces the symbol estimates lie close to the binary set.

We have proposed a computationally efficient multiuser detector that uses the nonlinear Gauss-Seidel iterations combined with the non-stationary iterated deregularization. It achieves a detection performance close to that of the optimal ML detector. Performance of the proposed detector, in terms of group detection error and worst-case complexity, has been investigated and compared with that of such advanced techniques as the decision-feedback detector, the branch-and-bound detector and the probabilistic data association detector. Our development significantly improves the "efficient frontier" in multiuser detection. The complexity of the proposed detector, approaching the optimal ML performance, is only $K^{3.2}$ FLOPS.

The performance of the proposed detector can be improved by using the optimal non-stationary deregularization parameter. However, such optimization is a complicated analytical problem that is the subject of a further work. Another problem to be solved relates to the fact that due to deregularization, the optimization problem may be non-convex, which also requires further investigation.

REFERENCES

[1] S. Verdu, *Multiuser Detection*, Cambridge University Press, 1998.

[2] J. Luo, K. R. Pattipati, P. K. Willett, and F. Hasegawa, "Near-optimal multiuser detection in synchronous CDMA using probabilistic data association," *IEEE Comm. Letters*, vol. 5, no. 9, pp. 361–363, Sept. 2001.

[3] J. Luo, K. Pattipati, P. Willett, and L. Brunel, "Branch-and-bound-based fast optimal algorithm for multiuser detection in synchronous CDMA," in *Proc. IEEE Int. Conf. on Communications, ICC'2003, Anchorage, Alaska, USA*, May 2003, vol. 5, pp. 3336–3340.

[4] J. Luo, K. R. Pattipati, P. Willet, and G. M. Levchuk, "Fast optimal and suboptimal any-time algorithms for CDMA multiuser detection based on branch and bound," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 632–641, April 2004.

[5] F. Hasegawa, J. Luo, K. Pattipati, P. Willett, and D. Pham, "Speed and accuracy comparison of techniques for multiuser detection in synchronous CDMA," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 540–545, April 2004.

[6] M. K. Varanasi, "Decision feedback multiuser detection: a systematic approach," *IEEE Trans. Inform. Theory*, vol. 45, pp. 219–240, Jan. 1999.

[7] A. Neumaier, "Solving ill-conditioned and singular linear systems: a tutorial on regularization," *SIAM Review*, vol. 40, no. 3, pp. 636–666, 1998.

[8] M. Hanke and C. W. Groetsch, "Nonstationary iterated Tikhonov regularization," *J. Optim. Theory Appl.*, vol. 98, pp. 37–53, 1998.

[9] C. Kasparis, R. J. Piechocki, P. N. Fletcher, and A. R. Nix, "A bootstrap multi-user detector for CDMA based on Tikhonov regularization," in *ICASSP'2003*, 2003.

[10] A. Yener, R. D. Yates, and S. Ulukus, "CDMA multiuser detection: a nonlinear programming approach," *IEEE Trans. Commun.*, vol. 50, no. 6, pp. 1016–1024, June 2002.

[11] L. K. Rasmussen, P. D. Alexander, and T. J. Lim, "A linear model for CDMA signals received with multiple antennas over multipath fading channels," in *CDMA Techniques for Third Generation Mobile Systems*, pp. 23–57. Kluwer Academic Publishers, 1999.

[12] M. A. Jensen and J. W. Wallace, "A review of antenna and propagation for MIMO wireless communications," *IEEE Trans. Antennas and Propagation*, vol. 52, no. 11, pp. 2810–2824, Nov. 2004.

[13] P. Spasojevic and C. N. Georghiades, "The slowest descent method and its application to sequence estimation," *IEEE Trans. Commun.*, vol. 49, no. 9, pp. 1592–1604, Sept. 2001.

[14] G. H. Golub and C. F. Van Loan, *Matrix computations*, The John Hopkins University Press, second edition, 1989.

[15] P. H. Tan and L. K. Rasmussen, "Linear interference cancellation in CDMA based on iterative techniques for linear equation systems," *IEEE Trans. Comm.*, vol. 48, no. 12, pp. 2099–2108, Dec. 2000.

[16] S. Chabbouh, P. Ciblat, and J.-C. Belfiore, "Iterative algorithms for SDP relaxation associated with MIMO ML detection problem," in *Int. Symp. Inf. Theory and its Applications, ISITA2004, Parma, Italy*, Oct. 10-13 2004.

[17] F. Albu, J. Kadlec, N. Coleman, and A. Fagan, "The Gauss-Seidel fast affine projection algorithm," in *Proceedings SIPS'2002, San Diego (CA)*, Oct. 2002, pp. 109–114.