Speed and Accuracy Comparison of Techniques to Solve a Binary Quadratic Programming Problem with Applications to Synchronous CDMA

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Abstract

In [11], it was shown that for solutions of the binary quadratic programming problem there exists an "Efficient Frontier" in the performance/speed domain among the algorithms which characterizes the relative performance of each algorithm. Here, in addition to the algorithms implemented in [11], Boltzmann machine, genetic algorithm and space alternating generalized EM (SAGE) receiver [22] have been implemented and results are given for much larger scale problems. Simulation results show that these and several other of the proposed methods can significantly outperform the decision feedback detector (DFD) or its group counterpart, GDFD.

1 Introduction

Binary quadratic programming (BQP) problems arise in a variety of applications, e.g., capital budgeting and financial analysis problems [13], [19], CAD problems [12], traffic message management problems [8], machine scheduling [1], molecular conformation and so on. Of particular current relevance, some digital communication problems, such as synchronous code-division multiple access (CDMA), can be formulated as BQPs.

In the CDMA context, prior research has focused on designing suboptimal receivers with low computational complexity and better performance than a conventional linear detector. Among them are the multistage detector [25], [26], the group detector [27] and the decision feedback detection [6], [7], [28]. Usually, suboptimal methods need to perform a projection to satisfy the integrality constraints, which can cause significant detection errors.

In this paper, we compare several methods for solving the BQP problem arising in synchronous CDMA. We discuss box-constrained quadratic programming, "best-first" and "depth-first" versions of branch and bound, coordinate descent, group decision-making, semi-definite relaxation, Probabilistic Data Association Filter (PDAF), SAGE receiver, genetic algorithm and Boltzmann machine. Simulation results show that the PDAF and several other of the proposed methods can significantly outperform the decision feedback detector (DFD) or its group counterpart, GDFD.

This paper is organized as follows. In section 2, the integer programming interpretation of synchronous CDMA problem is discussed. In section 3, various algorithms proposed for the integer programming problems are applied to the CDMA model, and are explained in detail. In section 4, simulation results and comparative analysis of the various algorithms are provided. Section 5 provides a summary and future research direction.

2 Problem Formulation

A discrete-time equivalent model for the matched-filter outputs at the receiver of a CDMA channel is given by the K-length vector [17]

$$\mathbf{y} = \mathbf{RW}b + n \tag{1}$$

where $b \in \{-1,+1\}^K$ denotes the K-length vector of bits transmitted by the K users. Here

$$\mathbf{H}=\mathbf{W}^{1/2}\mathbf{R}\mathbf{W}^{1/2} \tag{2}$$

is a positive definite signature waveform correlation matrix, $\mathbf{R}$ is the symmetric normalized correlation matrix with unit diagonal elements, $\mathbf{W}$ is a diagonal matrix whose k^{th} diagonal element, w_k is the received signal energy per bit of the k^{th} user, and n is a real-valued zero-mean Gaussian random vector with a covariance matrix $E[nn^T]=\sigma^2\mathbf{H}$. It has been shown that model (1) holds for both baseband [17] and passband [28] channels with additive white Gaussian noise.

When all the user signals are equally probable, the optimal solution of (1) is the output of a Maximum Likelihood (ML) detector [17]

$$\phi_{ML} : \hat{b} = \arg \min_{b \in \{-1,+1\}^K} \left(b^T \mathbf{H} b - 2y^T b\right) \tag{4}$$

The ML detector has the property that it minimizes, among all detectors, the probability that not all users' decisions are correct. Except in pathological cases, ϕ_{ML} is NP-hard and exponentially complex to implement; the focus is then on developing easily implementable integer programming algorithms for its solution.

3 CDMA Detection by Various Algorithms

3.1 Matched Filter

The simplest sub-optimal algorithm is a single-user matched filter. It makes a decision based on the sign of the observation in (1)

$$\phi_{MATCH}: \quad \hat{b}_M = sign(y) \tag{5}$$

It would be the optimal detector if the signature waveforms were orthogonal, i.e., if **H** were diagonal. This receiver is omitted from the plots in section 4 because of its poor performance.

3.2 Decorrelator

The conventional decorrelation detector improves on the matched filter output. It is found in two steps [17]. First, the unconstrained solution $\tilde{b} = \mathbf{H}^{-1} y$ is computed, and then this is projected onto the constrain set via: $\hat{b}_i = sign(b_i) \quad \forall i$. This receiver is also omitted from the plots in section 4, because of its poor performance

3.3 DFD Method

This method improves the probability of detection error by applying a successive cancellation technique on users. The DFD method based on the decorrelation detector, namely DDFD, is described in [28]. The users are sequentially demodulated by

$$\phi_{DFD}: \ \hat{b} = P\tilde{b}$$
$$\tilde{b}_i = sign\left(\sum_{j=1}^{K} F_{ij} P y_j - \sum_{j=1}^{i-1} B_{ij} \tilde{b}_j \right) \tag{6}$$

where **F**=U([**PHP**]$^{-1}$), **B**=L(**FPHP**). Here, $U(\cdot)$ represents the upper triangular part of a matrix, $L(\cdot)$ represents the strictly lower triangular part of a matrix, and **P** is a permutation matrix (symmetric and orthogonal). The choice of P has been discussed in Theorem 1 of [28]. Conceptually, the "easiest" (loosely, the most powerful) user, is detected first.

3.4 Quadratic Programming

The constraint in (4) can be relaxed to simple box constraints of the form as $-e \le b \le e$, where $b, e \in \Re^K$, and $e = [1 1 \cdots 1]^T$, with the understanding that the constraints be enforced at the final step via a hard-limiting projecting operation. Therefore, the minimization problem in (4) can be modeled as a box-constrained quadratic programming problem as follows:

$$\phi_{QP}: \quad \arg \min_{-1 \le b_i \le 1 \ \forall i} \left(b^T \mathbf{H} b - 2 y^T b \right) \tag{7}$$

Quadratic programs can be solved in a finite number of iterations; the positive definiteness of **H** and relaxed constraints allow the use of a variety of Newton-like methods to accelerate convergence. In our simulations, the Reflective Newton Method [3] is used to solve (7). This method is utilized in **quadprog()** of MATLAB 5.3 Optimization Toolbox to solve large-scale quadratic programming problems with box constraints. This algorithm was initialized with the output from the decorrelator.

3.5 Group Decision Feedback Detector (GDFD) with Optimal Grouping Algorithm

The idea of successive cancellation is that a correct decision on the strong users will improve the performance of weak users. In order to avoid an exponentially complex search among all users, it is intuitive to divide users into several groups; this was first introduced by Varanasi in [27], and significantly enhanced in [16]. In this simulation, the group size of the GDFD was set to 3. The details may be found in [16].

3.6 Semi-Definite Program

Semi-definite (SD) programs are convex optimization problems that unify several standard problems, e.g., linear and quadratic programming. It is shown in [18] that semi-definite relaxation is an accurate and efficient approximation method for certain NP-hard problems and it can approximately solve the Maximum-Likelihood Detection problem in $O(K^{3.5})$ complexity. In addition, since the model is convex, it does not suffer from local maxima. Moreover, efficient algorithms, namely interior-point methods, are available for solving the SD problem [24]. It is also shown that the SD relaxation solution can be converted to an approximate solution for a BQP problem by performing a computationally efficient randomization method [5], [21]. In this simulation, the number of the randomization was set to 20. The semidefinite model can be realized as follows:

$$\phi_{SD}: \quad (b^*, c^*) = \arg \max_{\substack{b \in \{-1,+1\}^K \\ c \in \{-1,+1\}}} \begin{bmatrix} b^T & c \end{bmatrix} \begin{bmatrix} -\mathbf{H} & \mathbf{W}^{\frac{1}{2}} y \\ \left(\mathbf{W}^{\frac{1}{2}} y\right)^T & 0 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} \tag{8}$$

where c is either –1 or 1. For the detailed algorithm, refer to [18].

3.7 Coordinate Descent Family

The problem in (4) is equivalent to the following [14]:

$$\phi_{CD}: \quad \hat{b} = \arg \min_{b \in \{0,1\}^K} \left(\tfrac{1}{2} x^T \mathbf{Q} x + c^T x \right) \tag{9}$$

where $x = \frac{b+e}{2}, c = (-y + \mathbf{H}e)$, $\mathbf{Q}$=2$\mathbf{H}$ and $e = [1 1 \cdots 1]^T$. The problem in (9) is inherently equivalent to

$$\phi_{CD}: \quad \hat{b} = \arg \min_{b \in \{0,1\}^K} \sum_{i<j} q_{ij} x_i x_j + \sum_{i=1}^{K} \left(c_i + \tfrac{1}{2} q_{ii} \right) x_i \tag{10}$$

where q_{ij} denotes the (i,j)[th] element of $\mathbf{Q}$. It is shown in [14] that by performing "greedy" local minimum search, a solution for (10) can be found. Both *Descent I* and *Descent II* algorithms, meaning we adjust one or two b_i's at each step respectively, are presented in [14].

3.8 Branch and Bound

The optimal solution to (4) can be obtained by interrogating each of the 2^K possible *b*'s. There are intelligent ways to compute such combinations. We introduce several variations of the branch and bound method in this section.

3.8.1 "Depth First" Branch and Bound

In, an optimal algorithm based on the branch and bound method with an iterative lower bound update was proposed. It was shown that the proposed method can significantly decrease the average computational cost.

3.8.2 "Best-First" Branch and Bound

The "best-first" search is a slightly different version of "depth-first" approach. As far as we know, this method is first applied to BQP here. The algorithm converges to an optimal solution. Several suboptimal variants can be derived by controlling the back tracking in the search process. For example, one could prematurely terminate the search when a feasible solution is found. For a detailed description of both "Depth-First" and "Best-First" Branch and Bound algorithm, refer to [11].

3.9 The Probabilistic Data Association Filter (PDAF) Approach

The PDAF is one of the simplest and most effective approaches to target tracking [2]. The PDAF idea can be applied to the CDMA model (1) as follows. The decisions on each user can be considered as binomial random variables, with the currently-estimated probabilities for the bit from user *i* to be +1 or -1, P_{bi} and 1- P_{bi}, respectively. By using a Gauss-Seidel iteration, the "soft" decisions are updated sequentially on all users. From (1), we have

$$y = r_i w_i b_i + \left(\sum_{j \neq i} r_j w_j b_j + n \right) \tag{11}$$

where r_i is the i^{th} column of $\mathbf{R}$ and w_i is the i^{th} diagonal element of $\mathbf{W}$. When updating the probability of user *i* to be +1 or -1, the combination of interferences from other users are considered approximately as a Gaussian random vector, with the mean and variance for user *j* calculated according to the current decision probabilities as $2P_{bj}$-1 and $4P_{bj}(1-P_{bj})$, respectively. We consequently obtain

$$P(y \mid b_i) \approx N\left(r_i w_i b_i + \left(\sum_{j \neq i} r_j w_j \left(2P_{b_j} - 1\right) \right), \sum_{j \neq i} 4P_{b_j}\left(1 - P_{b_j}\right) w_j^2 r_j r_j^T + \sigma^2 \mathbf{R} \right) \tag{12}$$

in which *N* refers to the standard normal probability density function (pdf) with the indicated mean and variance. Its computational burden is approximately $O(K^3)$, and as the simulation results show, its performance is nearly indistinguishable from that of the branch and bound detector. This algorithm is described in more detail in [11].

3.10 Genetic Algorithm

The implementation of a genetic algorithm (GA) to a CDMA application is proposed in [4]. It was shown that a hybrid approach, which combines a genetic algorithm with a multi-stage detector, yields the least probability of bit error among other variations of GA [4]. In this paper, the population size was set to $\frac{K}{3}$, where K is the number of users, and the number of generations was set to 20. Uniform-crossover operation was implemented and replacement type was Elitist. Crossover and mutation probability were set to 0.9 and 0.01, respectively.

3.11 Boltzmann Machine Algorithm

Motivated by the simplicity of the algorithm [10], the Boltzmann machine was applied to a CDMA application. The Boltzmann Machine is initialized with the DDFD output, and the "cost" for each case was calculated by evaluating (4). The following equation is used to make a decision for k^{th} user:

$$\phi_{BOLZ}: \quad \hat{b} = \max\left\{ \frac{e^{-\frac{q_{+1}}{T}}}{e^{-\frac{q_{+1}}{T}} + e^{-\frac{q_{-1}}{T}}}, \frac{e^{-\frac{q_{-1}}{T}}}{e^{-\frac{q_{+1}}{T}} + e^{-\frac{q_{-1}}{T}}} \right\} \qquad (13)$$

where q_{+1} denotes the cost from (4) when the k^{th} bit is set to +1. T is also to be decreased by a constant c, where $1 < c < \infty$. It is recommended that the initial temperature be set at a high value. In this simulation, it was set to 10^{12}. In the simulation, this algorithm was initialized with the output of DDFD.

3.12 SAGE Receiver

New iterative multiuser receivers based on the expectation-maximization (EM) algorithm and space alternating generalized EM algorithm (SAGE) were proposed in [22]. It was also shown in [22] that with the following *tanh* soft-decision decorrelator initialization, the SAGE receiver with hard decision could yield good performance.

$$b_k^0 = \tanh\left(\frac{a_k}{\left[\mathbf{R}^{-1}\right]_{kk} \sigma^2} \left[\mathbf{R}^{-1} y\right]_k \right) \quad k = 1, \ldots, K \qquad (14)$$

The above initialization method was used in the simulation.

4 Simulation Results

In this section, we compare the performance of the algorithms described in the previous section. The probability of group detection error is computed by varying the number of users with a fixed SNR at 15dB. The number of users tested are 40, 50, 60, 70, and 80. In each case, the signal energy for each user is generated such that $w_i \sim N(4.5,2)$ and $w_i \in [3,5]$ $\forall i$ - see equations (1), (2) and (4), and note that the language used here is that of the CDMA application. That is, any time the algorithmically-determined b does not match the solution to (4), this is termed a group detection error, and that the dimension of b is referred to as the number of users. The number of group detection errors are used to measure the accuracy of each algorithm based on 20000 Monte-Carlo runs. Important sampling was used to approximate the probability of error [9]. All of the simulations were run on a PC equipped with a Pentium III 600MHz processor. Additionally, the decision made by the PDAF detector is modified by flipping a bit sequentially to observe which, if any, can reduce the cost in (4). In other words, a local minimum is sought on the decision made by the PDAF detector. In the following figures, the relationship between the CPU time, which is normalized with the CPU time of DDFD, and the accuracy for each algorithm is shown. It is clear that DDFD, the "depth-first" branch and bound and PDAF form an "efficient frontier" for the algorithms. Using the accuracy and computations as performance meters, it is evident that the DDFD, PDAF and branch and bound algorithms dominate the remaining algorithms, viz., the matched filter, decorrelator, coordinate descent I and II, box-constrained quadratic programming method, semi-definite method, genetic algorithm, "best-first" branch and bound algorithm, SAGE receiver and Boltzmann machine algorithm. This, we believe, is a nice portrayal of the trade-off between accuracy and computational complexity.

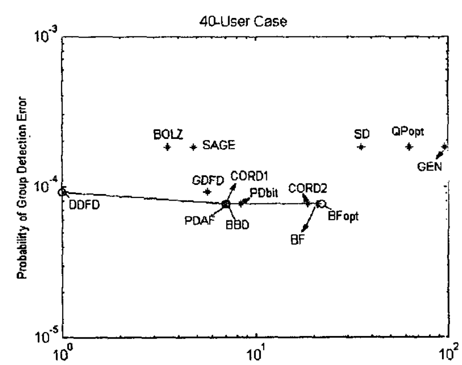

Figure 1: 40-User Case

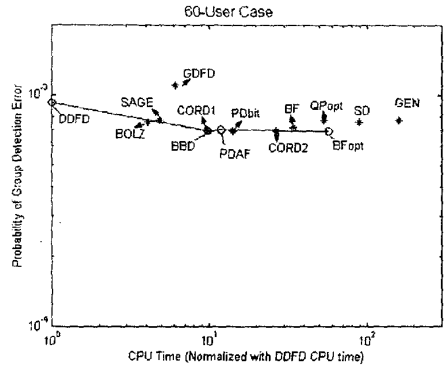

Figure 2: 60-User Case

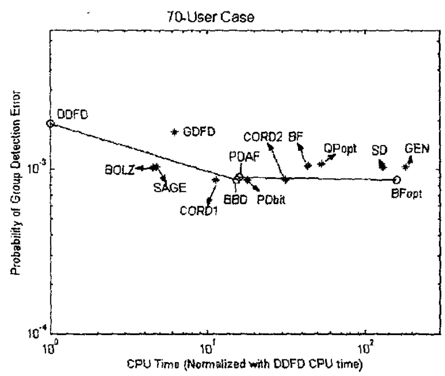

Figure 3: 70-User Case

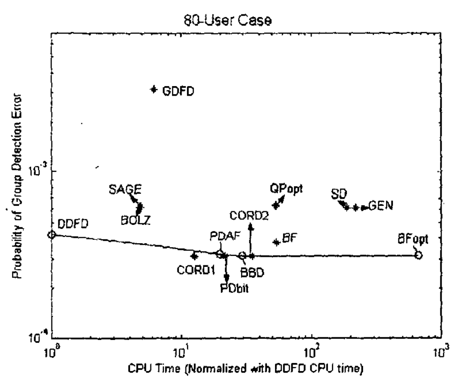

Figure 4: 80-User Case

5 Conclusion

As the simulation results show, most of the proposed algorithms are able to outperform the DDFD. The PDAF has exceptional performance among the other suboptimal methods. In addition, the same method is able to improve its accuracy slightly by performing bit-flip operations with $O(K^2)$ complexity. Other algorithms, namely the coordinate descent I and II, "depth-first" and "best-first" branch and bound methods, offer improved accuracy over DDFD, but at increased computational cost.

We have portrayed the trade-off between the accuracy and speed of the various approaches. From the results, we conclude that an "efficient frontier" is formed by DDFD, PDAF and branch and bound schemes; that is, all others are dominated by these algorithms. From Figure 1 through 4, it is clear that an inaccurate suboptimal method such as the GDFD cannot cross the "efficient frontier". Based on limited computational testing, accurate but time-consuming methods such as the coordinate descent II, or semi-definite programming or genetic algorithm appear not to cross the frontier. By observing these figures, it is clear that the PDAF is both time-efficient and accurate. Both SAGE receiver and Boltzmann machine algorithms were able to come near and cross the boundary. This again reveals the exceptional ability in both speed and accuracy of the PDAF detector, and the iterative methods, namely Boltzmann machine and SAGE receiver against the CDMA model.

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