

Modal Analysis Using Co-Prime Arrays

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Abstract—We address the problem of estimating mode parameters from noisy observations of a linear combination of the corresponding modes. This problem arises in line spectrum estimation, vibration analysis, speech processing, system identification, and direction of arrival estimation. Our results differ from standard results of modal analysis to the extent that we consider co-prime samplings in space, or equivalently co-prime samplings in time. Our main result is a characterization of the orthogonal subspace for this problem. This is the subspace that is orthogonal to the signal subspace spanned by the columns of the generalized Vandermonde matrix of modes in co-prime samplings. This characterization is derived in a form that allows us to adapt modern methods of subspace signal processing to co-prime sampled signals. Several numerical examples are presented to demonstrate the application of the proposed modal estimation method. We state and prove theorems on identifiability of the modes and calculate a Cramér-Rao bound that allows us to analyze the performance of co-prime arrays that are subsamplings of uniform linear arrays of the same apertures.

Index Terms—Co-prime array, Cramér-Rao bound, modal analysis, orthogonal subspaces.

I. INTRODUCTION

IN this paper, we investigate the problem of estimating the parameters of damped complex exponential modes from the observation of co-prime samples of their weighted sum. This problem arises in many applications such as spectrum analysis, speech processing, system identification, and direction of arrival (DOA) estimation.

There is a vast literature on modal estimation methods from uniformly sampled time or space series data, starting with the work of Prony [6]. Other methods include approximate least squares or maximum likelihood estimation [7], [8], reduced rank linear prediction [9], [10], MUSIC [11], and ESPRIT [12]. While there are extensions of MUSIC and ESPRIT for

direction of arrival estimation from non-uniformly sampled data (see, e.g., [13]–[17]), Prony-like methods have mainly been developed for uniformly sampled data, and extending such methods to non-uniformly sampled data has not received much attention.¹

Non-uniform sensor array geometries, without aliasing ambiguities, have a long history in sensor array processing, dating back to minimum-redundancy arrays [20]. The introduction and analysis of nested arrays in [21] and co-prime arrays in [1]–[5] have created renewed interest in such geometries. A co-prime array consists of two uniform subarrays, where interelement spacings (in units of half-wavelength) of the two subarrays are co-prime. The co-prime property allows for the resolution of aliasing ambiguities as reported in [1]–[5].

Measurements in a co-prime array may be used directly for inference, or they may be used indirectly to fill in the entries of a Toeplitz covariance matrix in a difference co-array in a *second-order* measurement model [1]–[5]. The direct case has been studied in [3], [4], where the authors establish identifiability theorems for identifying undamped modes, with non-singular covariance matrix of the mode amplitudes, using MUSIC.² The indirect methodology has been studied in [1], [2], and [5], and applies when the modes to be identified are undamped, and mode amplitudes are uncorrelated. Under these assumptions, the authors of [1], [2], and [5] prove theorems that show that $\mathcal{O}(m^2)$ sources may be identified from m sensor elements. These are *identifiability* results. In other words, there is no consideration of model fitting to a sample covariance matrix computed from a finite number of noisy samples. The caution here is that these identifiability theorems speak only to well-posedness of models. They do not speak to sensitivity to model mismatch or to performance when a model is to be identified from a sequence of noisy array snapshots.

Performance (in white noise) of co-prime arrays has been studied and compared with that of a ULA in [3] and [4], for the case where the number of sensors in the co-prime array equals the number of sensors in the ULA, but the aperture of the co-prime array is much larger than the aperture of the ULA. In this case, the Cramér-Rao bound (CRB) for DOA estimation favors the co-prime array. This analysis is relevant in applications where a long aperture is feasible (e.g., in radio astronomy).

In this paper, we derive *identifiability* theorems and algorithms for modal analysis with co-prime arrays, in a *first-order* measurement model. Obviously, difference co-array processing

¹In [18] and [19], the authors develop interpolation methods for approximating uniformly spaced samples from non-uniformly taken samples. Prony-like methods are then used with the interpolated samples for modal analysis. These methods are quite different from methods and results reported in this paper, and are not addressed to co-prime sampling.

²These results can easily be generalized to include damped modes.

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is ruled out and our approach uses the co-prime array data directly. Our assumptions are different from those of [1]–[5]. In the model considered in this paper, modes may be damped or undamped, mode amplitudes need not be uncorrelated, and snapshots are not necessarily plentiful. In fact we may have only a single snapshot. Performance is bounded with CRBs for mode parameters.

We consider two specific cases of co-prime sensor arrays. Both of these geometries may be viewed as subsampled versions of a dense uniform line array of the *same* aperture, with uniform half wavelength spacings.³ Specifically, we first consider a special co-prime array, which we call the *restricted co-prime array*. This array may be thought of as a uniformly subsampled version of a dense uniform line array, plus an extra sensor that is positioned at a location on the array that allows us to resolve aliasing ambiguities. Then we consider the conventional co-prime array which consists of two uniform subarrays, each obtained by uniformly subsampling a dense uniform line array with co-prime subsampling factors.

In each case, we determine a parameterization of a subspace that is orthogonal to the signal subspace spanned by the columns of a generalized Vandermonde matrix of the modes in a co-prime array. This parameterization is of a form that is particularly suitable for utilizing methods based on subspace signal processing (see [7]–[12], and [22]), for estimating the modes.

For performance analysis, we study CRBs, error concentration ellipses, and mean-squared errors (MSEs) in the case where the aperture of the co-prime array is specified to equal the aperture of a ULA, but many fewer sensors are used in the co-prime array than would be used in the ULA. This is the typical case in radar and sonar, where the limiting factors are aperture and observation time. Although we present our numerical results in the context of sensor array processing, all of our results apply to the estimation of complex exponential modes from time series data.

Remark 1: Naturally, for a fixed aperture, any subsampling in space results in a reduction in effective signal-to-noise ratio (SNR), by the subsampling factor, and leads to a loss in estimation performance. Our studies in [23]–[25] address the effects of compression and subsampling on Fisher information, the Cramér-Rao bound, and the probability of a swap between signal and noise subspaces. Assuming that the loss in SNR due to subsampling has tolerable effects on estimation or detection, the question is how can subspace methods of linear prediction be adapted to the estimation of mode parameters in a co-prime array?

II. PROBLEM STATEMENT

Consider a non-uniform line array of m sensors at locations $\mathbb{l} = \{i_0, i_1, \dots, i_{m-1}\}$ in units of half wavelength in space. We assume, without loss of generality, that $i_0 = 0$. Suppose the array is observing a weighted superposition of p damped complex exponential modes. These modes are determined by the mode parameters $z_k = \rho_k e^{j\theta_k}$, $k = 1, 2, \dots, p$, where the

³If we were sampling in time, then the dense sequence of uniform samples would have had spacings equal to the Nyquist interval.

k th mode has a damping factor $\rho_k \in \mathbb{R}^+$ and an electrical angle $\theta_k \in (-\pi, \pi]$. Suppose the array collects N temporal snapshots. Then, the measurement equation for the l th sensor (located at i_l) may be written as

$$y_l[n] = \sum_{k=1}^p x_k[n] z_k^{i_l} + e_l[n], \quad n = 0, 1, \dots, N-1, \quad i_l \in \mathbb{l}, \quad (1)$$

where n is the snapshot index, $x_k[n]$ denotes the amplitude (or weight) of the k th mode at index n , and $e_l[n]$ is the measurement noise at sensor l . In vector form, we have $\mathbf{y}[n] \in \mathbb{C}^m$,

$$\mathbf{y}[n] = \mathbf{V}(\mathbf{z}, \mathbb{l}) \mathbf{x}[n] + \mathbf{e}[n], \quad n = 0, 1, \dots, N-1, \quad (2)$$

where $\mathbf{y}[n] = [y_0[n], y_1[n], \dots, y_{m-1}[n]]^T$ is the array measurement vector, $\mathbf{x}[n] = [x_1[n], x_2[n], \dots, x_p[n]]^T$ is the vector of mode amplitudes at index n , $\mathbf{e}[n] = [e_0[n], e_1[n], \dots, e_{m-1}[n]]^T$ is the noise vector at index n , and $\mathbf{V}(\mathbf{z}, \mathbb{l}) = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_p, \mathbb{l})] \in \mathbb{C}^{m \times p}$ is a generalized Vandermonde matrix of the modes $\mathbf{z} = [z_1, z_2, \dots, z_p]^T$, given by

$$\mathbf{V}(\mathbf{z}, \mathbb{l}) = \begin{bmatrix} z_1^{i_0} & z_2^{i_0} & \dots & z_p^{i_0} \\ z_1^{i_1} & z_2^{i_1} & \dots & z_p^{i_1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{i_{m-1}} & z_2^{i_{m-1}} & \dots & z_p^{i_{m-1}} \end{bmatrix}. \quad (3)$$

Our view is that modal analysis is addressed to the identification of physical mode parameters $\{\rho_k \in \mathbb{R}^+, -\pi \leq \theta_k \leq \pi\}_{k=1}^p$. According to any reasonable measure on the complex plane, the probability that any such set would produce a matrix $\mathbf{V}(\mathbf{z}, \mathbb{l})$ that is rank deficient is zero, for proper choice of sensor locations. Therefore, we assume throughout that modes are not so specially drawn that the sampling pattern leaves $\mathbf{V}(\mathbf{z}, \mathbb{l})$ rank deficient. See [26] for a discussion of this issue.

We consider the case where $\mathbf{x}[n]$ is free to change with n , and assume that the $e_l[n]$'s are i.i.d. proper complex normal with mean zero and variance σ^2 . This means that the measurement vectors $\mathbf{y}[n]$, $n = 0, 1, \dots, N-1$ are i.i.d. proper complex normal with mean $\mathbf{V}(\mathbf{z}, \mathbb{l}) \mathbf{x}[n]$ and covariance $\sigma^2 \mathbf{I}$. Under this measurement model, the least squares estimation and the maximum likelihood estimation of the modes $\{z_k\}_{k=1}^p$ and mode weights $\{\mathbf{x}[n]\}_{n=0}^{N-1}$ are equivalent and may be posed as

$$\min_{\mathbf{z}, \mathbf{x}[0], \dots, \mathbf{x}[N-1]} \sum_{n=0}^{N-1} \|\mathbf{y}[n] - \mathbf{V}(\mathbf{z}, \mathbb{l}) \mathbf{x}[n]\|_2^2. \quad (4)$$

The least squares estimate of $\mathbf{x}[n]$ is

$$\hat{\mathbf{x}}[n] = \mathbf{V}^+(\mathbf{z}, \mathbb{l}) \mathbf{y}[n], \quad (5)$$

where $\mathbf{V}^+(\mathbf{z}, \mathbb{l}) = (\mathbf{V}^H(\mathbf{z}, \mathbb{l}) \mathbf{V}(\mathbf{z}, \mathbb{l}))^{-1} \mathbf{V}^H(\mathbf{z}, \mathbb{l})$ is the Moore-Penrose pseudoinverse of $\mathbf{V}(\mathbf{z}, \mathbb{l})$. The least squares estimate of the modes is obtained as (see [27], pp. 245–246)

$$\begin{aligned} \hat{\mathbf{z}} &= \arg \min_{\mathbf{z}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] (\mathbf{I} - \mathbf{P}_{\mathbf{V}(\mathbf{z}, \mathbb{l})}) \mathbf{y}[n] \\ &= \arg \min_{\mathbf{z}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_{\mathbf{A}(\mathbf{z}, \mathbb{l})} \mathbf{y}[n]. \end{aligned} \quad (6)$$

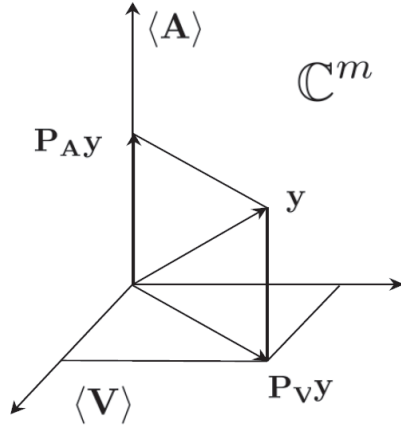


Fig. 1. The signal subspace $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ and the orthogonal subspace $\langle \mathbf{A}(\mathbf{z}, \mathbb{l}) \rangle = \langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle^\perp$. In the figure, we have dropped (\mathbf{z}, \mathbb{l}) and have simply used \mathbf{A} , \mathbf{V} , $\langle \mathbf{A} \rangle$, and $\langle \mathbf{V} \rangle$.

Here $\mathbf{A}(\mathbf{z}, \mathbb{l})$ is a full column rank matrix that satisfies

$$\mathbf{A}^H(\mathbf{z}, \mathbb{l})\mathbf{V}(\mathbf{z}, \mathbb{l}) = \mathbf{0}_{(m-p) \times p}, \quad (7)$$

and the projection matrices $\mathbf{P}_{\mathbf{V}(\mathbf{z}, \mathbb{l})}$ and $\mathbf{P}_{\mathbf{A}(\mathbf{z}, \mathbb{l})} = \mathbf{I} - \mathbf{P}_{\mathbf{V}(\mathbf{z}, \mathbb{l})}$ are the orthogonal projections onto the column spans of $\mathbf{V}(\mathbf{z}, \mathbb{l})$ and $\mathbf{A}(\mathbf{z}, \mathbb{l})$, respectively. We denote these column spans by the subspaces $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ and $\langle \mathbf{A}(\mathbf{z}, \mathbb{l}) \rangle$. We call $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ the *signal subspace* and $\langle \mathbf{A}(\mathbf{z}, \mathbb{l}) \rangle$ the *orthogonal subspace*. We assume throughout this paper that $\mathbf{V}(\mathbf{z}, \mathbb{l})$ is a full column rank matrix, thus $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ has dimension p and $\langle \mathbf{A}(\mathbf{z}, \mathbb{l}) \rangle$ has dimension $m - p$. Therefore, the nullspace of $\mathbf{A}^H(\mathbf{z}, \mathbb{l})$ has dimension p . Equivalently, $\langle \mathbf{A}(\mathbf{z}, \mathbb{l}) \rangle = \langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle^\perp$. See Fig. 1.

It is important to note that this deterministic least squares formulation leads to the objective function (6) for the mode parameters $\mathbf{z} = [z_1, \dots, z_p]^T$, but it does not introduce the mode weights $(\mathbf{x}[n], n = 0, \dots, N - 1)$ into the objective function. So it is the maximum likelihood or least squares objective function for the mode parameters $\mathbf{z} = [z_1, \dots, z_p]^T$ in a joint least squares estimation $(\mathbf{z}, \mathbf{x}[0], \dots, \mathbf{x}[N - 1])$. If we were interested in the mode weights, they could be estimated using (5). When the amplitudes $\mathbf{x}[n], n = 0, \dots, N - 1$ are given a multivariate normal probability law, then these parameters are marginalized out and replaced by their covariance, resulting in a stochastic rather than deterministic least squares problem.

Theorem 1: Let $\mathbf{S}_{xx} = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{x}[n]\mathbf{x}^H[n] \succ \mathbf{0}$, and $\mathbf{e}[n] = \mathbf{0}$, for $n = 0, 1, \dots, N - 1$, in (2). Then the least squares problem in (6) uniquely identifies any p distinct modes, if and only if the $m \times (p + 1)$ matrix $\mathbf{V}_{aug} = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_p, \mathbb{l}), \mathbf{v}(z_{p+1}, \mathbb{l})]$ has full column rank $p + 1$ for all possible distinct modes $\{z_k\}_{k=1}^{p+1}$. Thus $p \leq m - 1$.

Proof: See Appendix A for a proof in the style of [28].

Remark 2: In the proof of Theorem 1, the number of modes, p , is less than or equal to $m - 1$. Moreover, for $\mathbf{S}_{xx} \succ \mathbf{0}$, the number of snapshots, N , is greater than or equal to p . So, $p \leq \min\{m - 1, N\}$. However, the proof may be modified for a single snapshot $\mathbf{y}[0] = \mathbf{V}(\mathbf{z}, \mathbb{l})\mathbf{x}[0]$, provided that no element of

$\mathbf{x}[0]$ is zero, meaning no coordinate of the subspace $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ is excluded in the construction of $\mathbf{y}[0]$. This condition holds with probability one, because for any reasonable probability measure on the complex plane the probability that a mode amplitude is zero is equal to zero. In this case, the least squares problem in (6) uniquely identifies any p distinct modes, if and only if the $m \times 2p$ matrix $\mathbf{V}_{aug} = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_{2p}, \mathbb{l})]$ has full column rank $2p$ for all possible distinct modes $\{z_k\}_{k=1}^{2p}$. Thus $2p \leq m$. A sketch of the proof for this case is given in a paragraph after the proof of Theorem 1 in Appendix A. Naturally, the least squares problem in (6) uniquely identifies any p distinct modes from $1 \leq N < p$ snapshots, if the mode amplitudes of at least one of the snapshots has no zero element, and the $m \times 2p$ matrix $\mathbf{V}_{aug} = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_{2p}, \mathbb{l})]$ has full column rank $2p$ for all possible distinct modes $\{z_k\}_{k=1}^{2p}$. Thus $2p \leq m$.

For a given array geometry, the *basis matrix* $\mathbf{V}(\mathbf{z}, \mathbb{l})$ given in (3) and the subspace $\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle$ are fully characterized by the p modes $\mathbf{z} = [z_1, z_2, \dots, z_p]^T$. This subspace, parameterized by \mathbf{z} , is an element of a Grassmanian manifold $\mathcal{G}(p, m)$. However, it is not easy to solve the least squares problem (6) using this characterization. Alternatively, for an m -element *uniform* line array, a particular p -parameter characterization of $\mathbf{A}(\mathbf{z}, \mathbb{l})$ exists that makes solving (6) relatively simple [6]. We will review this characterization in Section III. Then, we derive parameterizations of $\mathbf{A}(\mathbf{z}, \mathbb{l})$ for two specific array geometries: restricted co-prime and co-prime.

- *Restricted co-prime array:* In this case, the location set \mathbb{l} is given by $\mathbb{l}_s = \{0, d, 2d, \dots, (m - 2)d, M\}$, where M and d are co-prime integers, that is, $(M, d) = 1$, and $d > 1$. This array may be thought of as two subarrays. The first is a subsampled version, by a factor d , of an $(m - 1)d$ -element uniform line array (ULA) with half wavelength inter-element spacings. The second is a single sensor at location M in the line array such that M and d are co-prime. We note that M need not be greater than $(m - 2)d$.
- *Co-prime array:* In this case, $\mathbb{l} = \mathbb{l}_1 \cup \mathbb{l}_2$, where $\mathbb{l}_1 = \{0, m_2, 2m_2, \dots, (m_1 - 1)m_2\}$, $\mathbb{l}_2 = \{m_1, 2m_1, \dots, (2m_2 - 1)m_1\}$, and $(m_1, m_2) = 1$. Again the array is composed of two subarrays. The first is an m_1 -element ULA with interelement spacings of m_2 and sensor locations \mathbb{l}_1 . The second is a $(2m_2 - 1)$ -element ULA with interelement spacings of m_1 and sensor locations \mathbb{l}_2 . This co-prime geometry was introduced in [1] and [2].

Remark 3: In both cases above, the co-prime constraint guarantees that aliasing ambiguities due to subsampling can be resolved [3], [4]. This is due to the *invertibility of the array manifold* for co-prime arrays (see Theorem 1 of [4]). Although a restricted co-prime array can be viewed as a special case of a conventional co-prime array, we consider them separately, because it is easier to first derive a suitable characterization of the orthogonal subspace $\langle \mathbf{A} \rangle$ for a restricted co-prime array, and then generalize it to a co-prime array. Our parameterizations are not minimal. They involve $2p$ parameters, instead of p , but as we will show in Section IV, they are specifically designed to utilize modern subspace methods of modal analysis.

Remark 4: By now it should be clear that $\mathbf{A}(\mathbf{z}, \mathbb{l})$, and therefore its parameterization, depend on both the mode vector \mathbf{z} and

the array geometry $\mathbb{1}$. Therefore, from here on, we may drop $(\mathbf{z}, \mathbb{1})$ and simply use \mathbf{A} , \mathbf{V} , $\langle \mathbf{A} \rangle$, and $\langle \mathbf{V} \rangle$.

III. CHARACTERIZATION OF THE ORTHOGONAL SUBSPACE FOR UNIFORM LINE ARRAYS

Consider a uniform line array of m equidistant sensors located at $\mathbb{1}_u = \{0, 1, 2, \dots, m-1\}$, taking measurements from the superposition of p modes as in (2). The signal subspace in this case is characterized by the Vandermonde matrix \mathbf{V} in (3) with $\mathbb{1} = \mathbb{1}_u$. Assume $z_i \neq z_j$ for $i \neq j$. To characterize the orthogonal subspace $\langle \mathbf{A} \rangle$, consider the polynomial $A(z)$:

$$\begin{aligned} A(z) &= \prod_{k=1}^p (1 - z_k z^{-1}) \\ &= \sum_{i=0}^p a_i z^{-i}; \quad a_0 = 1 \end{aligned} \quad (8)$$

which has (z_1, z_2, \dots, z_p) as its p complex roots. The $(m-p)$ dimensional orthogonal subspace $\langle \mathbf{A} \rangle$ is spanned by the $m-p$ linearly independent columns of \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} a_p & a_{p-1} & \cdots & a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_p & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & a_p & \cdots & \cdots & a_1 & 1 \end{bmatrix}^H. \quad (9)$$

Since $\mathbf{A}^H \mathbf{V} = \mathbf{0}$, and the columns of \mathbf{A} are linearly independent, \mathbf{V} and \mathbf{A} span orthogonal subspaces $\langle \mathbf{V} \rangle$ and $\langle \mathbf{A} \rangle$ in \mathbb{C}^m . The above parameterization is at the heart of subspace methods of modal analysis such as IQML, MUSIC and ESPRIT (see, e.g., [6]–[12]).

Using the p -parameter representation for $\langle \mathbf{A} \rangle$ in (9) we may re-write the least squares problem of (6) as

$$\hat{\mathbf{a}} = \underset{\mathbf{a}=[a_1, \dots, a_p]^T \in \mathbb{C}^p}{\operatorname{argmin}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_A \mathbf{y}[n]. \quad (10)$$

The polynomial $\hat{A}(z) = 1 + \sum_{i=1}^p \hat{a}_i z^{-i}$ is then formed from the estimate $\hat{\mathbf{a}}$ in (10), and its roots are taken as the mode estimates $(\hat{z}_1, \dots, \hat{z}_p)$.

IV. CHARACTERIZATION OF THE ORTHOGONAL SUBSPACES FOR CO-PRIME ARRAYS

In this section, we present simple characterizations of the orthogonal subspace $\langle \mathbf{A} \rangle$ for the co-prime arrays discussed in Section II. Our characterizations can be exploited to adapt subspace methods of modal analysis to estimate the complex exponential modes in such arrays.

A. Restricted Co-Prime Array

Consider the restricted co-prime array described in Section II. The set of sensor locations for this array is

$\mathbb{1}_s = \{0, d, 2d, \dots, (m-2)d, M\}$. The generalized Vandermonde matrix \mathbf{V} in this case is

$$\mathbf{V}(\mathbf{z}, \mathbb{1}_s) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1^d & z_2^d & \cdots & z_p^d \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(m-2)d} & z_2^{(m-2)d} & \cdots & z_p^{(m-2)d} \\ z_1^M & z_2^M & \cdots & z_p^M \end{bmatrix}. \quad (11)$$

We assume that $p < m-1$ and that the modes $\{z_k\}_{k=1}^p$ are such that $z_i^d \neq z_j^d$ for $i \neq j$. In [4], this assumption is made for the undamped modes, and it guarantees identifiability of undamped modes from the restricted co-prime array measurements using MUSIC. In our problem, this assumption guarantees the identifiability of the p damped or undamped modes from our approximate least squares algorithm, as will be shown in this section. Also, for such a set of modes the top $m-1$ rows of $\mathbf{V}(\mathbf{z}, \mathbb{1}_s)$ form a full rank Vandermonde matrix, thus $\mathbf{V}(\mathbf{z}, \mathbb{1}_s)$ has rank p .

For $d > 1$ it is clear that without the use of the last sensor at location M , we cannot unambiguously estimate the modes, because any two modes z_k and $z_k e^{j2\pi q/d}$, $q = 1, 2, \dots, d-1$ produce the same measurement. This is the aliasing problem for subsampled arrays.

To characterize the $(m-p)$ -dimensional orthogonal subspace $\langle \mathbf{A} \rangle$, determined by the modes $\{z_k\}_{k=1}^p$, we first form the polynomial $A(z)$ from the d th powers of z_k , namely the $w_k = z_k^d$, $k = 1, 2, \dots, p$:

$$A(z) = \prod_{k=1}^p (1 - w_k z^{-1}) = \sum_{i=0}^p a_i z^{-i}; \quad a_0 = 1. \quad (12)$$

Since $\{w_k\}_{k=1}^p$ are the roots of $A(z)$, the first $m-p-1$ columns \mathbf{A}_0 of $\mathbf{A} \in \mathbb{C}^{m \times (m-p)}$, which is to satisfy $\mathbf{A}^H \mathbf{V} = \mathbf{0}$, can be written as

$$\mathbf{A}_0 = \begin{bmatrix} a_p & a_{p-1} & \cdots & a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_p & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & a_p & \cdots & \cdots & a_1 & 1 & 0 \end{bmatrix}^H. \quad (13)$$

But of course any mode of the form $z_k e^{j2\pi q/d}$, $q = 1, \dots, d-1$, would produce the same w_k and therefore the same \mathbf{A}_0 . This is the ambiguity caused by aliasing.

Now, consider the polynomial

$$B(z) = z^M + \sum_{i=1}^p b_i z^{(p-i)d}. \quad (14)$$

Suppose the coefficient vector $\mathbf{b} = [b_1, b_2, \dots, b_p]^T$ is such that the actual modes $\{z_k\}_{k=1}^p$ are the roots of $B(z)$. That is, $B(z_k) = 0$ for $k = 1, 2, \dots, p$. Then, since M and d are co-prime, for $1 \leq q \leq d-1$ and $1 \leq k \leq p$ we have

$$\begin{aligned} B(z_k e^{j2\pi q/d}) &= z_k^M e^{j2\pi M q/d} + \sum_{i=1}^p b_i z_k^{(p-i)d} \\ &= z_k^M (e^{j2\pi M q/d} - 1) \\ &\neq 0 \quad \text{for } q = 1, 2, \dots, d-1. \end{aligned} \quad (15)$$

Therefore, the only common roots of $B(z)$, and the d th roots of $\{w_k\}_{k=1}^p$, are $\{z_k\}_{k=1}^p$, which are the actual modes to be estimated. In this way, $B(z)$ resolves the ambiguities.

Now suppose $\{w_k\}_{k=1}^p$ are known (or estimated). Then from (14), \mathbf{b} can be found by solving the linear system of equations

$$(b_p \ b_{p-1} \ \cdots \ b_1) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1^d & z_2^d & \cdots & z_p^d \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(p-1)d} & z_2^{(p-1)d} & \cdots & z_p^{(p-1)d} \end{pmatrix} = - (z_1^M \ z_2^M \ \cdots \ z_p^M), \quad (16)$$

which if $z_i^d \neq z_j^d$ for $i \neq j$ (as we assume) has a unique solution.

Using the $2p$ coefficients $\{a_i\}_{i=1}^p$ and $\{b_i\}_{i=1}^p$, we can characterize $\langle \mathbf{A} \rangle$ by writing $\mathbf{A} \in \mathbb{C}^{m \times (m-p)}$ as

$$\mathbf{A} = \begin{bmatrix} a_p & a_{p-1} & \cdots & a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_p & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & a_p & \cdots & a_1 & 1 & 0 \\ b_p & b_{p-1} & \cdots & b_1 & 0 & \cdots & 0 & 1 \end{bmatrix}^H. \quad (17)$$

From the structure of \mathbf{A} in (17), it is obvious that the columns of the \mathbf{A} are linearly independent, therefore \mathbf{A} has rank $m - p$, and the dimension of its nullspace is p .

To estimate $\mathbf{a} = [a_1, \dots, a_p]^T$ and $\mathbf{b} = [b_1, \dots, b_p]^T$, we solve the following problem:

$$\min_{\mathbf{a}, \mathbf{b}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_{\mathbf{A}} \mathbf{y}[n]. \quad (18)$$

We approximate the solution to this problem in two steps. First, we ignore the last row of \mathbf{A}^H and estimate $\hat{\mathbf{a}}$ as

$$\hat{\mathbf{a}} = \underset{\mathbf{a}}{\operatorname{argmin}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_{\mathbf{A}_0} \mathbf{y}[n]. \quad (19)$$

The minimization problem in (19) can be solved using any subspace method of modal analysis. Now, given $\hat{\mathbf{a}}$, we form the polynomial

$$\begin{aligned} \hat{A}(z) &= 1 + \sum_{i=1}^p \hat{a}_i z^{-i} \\ &= \prod_{k=1}^p (1 - \hat{w}_k z^{-1}) \end{aligned} \quad (20)$$

and derive its roots as $\{\hat{w}_k\}_{k=1}^p$. But we know from the structure of the problem that $\hat{w}_k = \hat{z}_k^d$, and any of the d -th roots of \hat{z}_k^d is a candidate solution. Therefore, we construct the candidate set,

$$\mathcal{R} = \left\{ \left(\hat{z}_1 e^{j2\pi q_1/d}, \hat{z}_2 e^{j2\pi q_2/d}, \dots, \hat{z}_p e^{j2\pi q_p/d} \right) \mid 0 \leq q_1, q_2, \dots, q_p \leq d-1 \right\}, \quad (21)$$

which contains all modes and their aliased versions.

In the second step, to find the p actual modes and resolve aliasing ambiguities, we solve the following constrained linear prediction problem:

$$\begin{aligned} \hat{\mathbf{b}} &= \underset{\boldsymbol{\zeta}}{\operatorname{argmin}} \sum_{n=0}^{N-1} |y_{m-1}[n] + \boldsymbol{\zeta}^T \mathbf{u}[n]|^2 \\ \text{s.t. } & B_{\boldsymbol{\zeta}}(\hat{\mathbf{z}}) = 0, \quad \hat{\mathbf{z}} \in \mathcal{R}, \end{aligned} \quad (22)$$

where $\mathbf{u}[n] = [y_0[n], y_1[n], \dots, y_{p-1}[n]]^T$, and the polynomial $B_{\boldsymbol{\zeta}}(z)$ is obtained from replacing \mathbf{b} by $\boldsymbol{\zeta}$ in (14).

The procedure for estimating modes in a restricted co-prime array is summarized in Algorithm 1.

Algorithm 1: Modal Estimation from the Restricted Co-Prime Array

1: Estimate $\hat{\mathbf{a}} = [\hat{a}_1, \hat{a}_2, \dots, \hat{a}_p]^T$ from the least squares problem in (19). That is,

$$\hat{\mathbf{a}} = \underset{\mathbf{a}}{\operatorname{argmin}} \sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_{\mathbf{A}_0} \mathbf{y}[n],$$

where \mathbf{A}_0 is given in (13);

2: Root $\hat{A}(z)$ in (20) to return roots $\{\hat{w}_k\}_{k=1}^p$. Then, recognizing that the d th roots of \hat{w}_k are $\hat{z}_k e^{j2\pi q/d}$ for some $q \in \{0, 1, 2, \dots, d-1\}$, form the set of candidate modes \mathcal{R} as in (21). That is,

$$\mathcal{R} = \left\{ \left(\hat{z}_1 e^{j2\pi q_1/d}, \hat{z}_2 e^{j2\pi q_2/d}, \dots, \hat{z}_p e^{j2\pi q_p/d} \right) \mid 0 \leq q_1, q_2, \dots, q_p \leq d-1 \right\};$$

3: Solve (22) for $\hat{\mathbf{b}}$. That is,

$$\begin{aligned} \hat{\mathbf{b}} &= \underset{\boldsymbol{\zeta}}{\operatorname{argmin}} \sum_{n=0}^{N-1} |y_{m-1}[n] + \boldsymbol{\zeta}^T \mathbf{u}[n]|^2 \\ \text{s.t. } & B_{\boldsymbol{\zeta}}(\hat{\mathbf{z}}) = 0, \quad \hat{\mathbf{z}} \in \mathcal{R}; \end{aligned}$$

4: Intersect the roots of $\hat{B}(z) = z^M + \sum_{i=1}^p \hat{b}_i z^{(p-i)d}$ with \mathcal{R} .

Theorem 2: Let $\mathbf{S}_{xx} \succ \mathbf{0}$, and $\mathbf{e}[n] = \mathbf{0}$, for $n = 0, 1, \dots, N-1$, in (2). Consider a restricted co-prime array with the set of sensor locations defined as $\mathbb{I}_s = \{0, d, 2d, \dots, (m-2)d, M\}$ for $(d, M) = 1$. Let there be p modes $\{z_k\}_{k=1}^p$, such that $z_i^d \neq z_j^d$ for $i \neq j$, and $p \leq m-2$. Then, Algorithm 1 uniquely identifies these modes from the restricted co-prime array specified by \mathbb{I}_s .

Proof: See Appendix B.

Remark 5: Similar to Theorem 1, Theorem 2 requires $p \leq \min\{m-2, N\}$. However, the proof may be modified to hold for a single snapshot $\mathbf{y}[0] = \mathbf{V}(\mathbf{z}, \mathbb{I})\mathbf{x}[0]$, provided that no element of $\mathbf{x}[0]$ is zero, meaning no coordinate of the subspace $\langle \mathbf{V}(\mathbf{z}, \mathbb{I}) \rangle$ is excluded in the construction of $\mathbf{y}[0]$. In the single snapshot case, p modes $\{z_k\}_{k=1}^p$ ($z_i^d \neq z_j^d$ for $i \neq j$) can be identified provided that $2p \leq m-1$. The proof follows from a

straightforward modification of the proof in Appendix B. Naturally, Theorem 2 still holds for $1 < N < p$ snapshots, if the mode amplitudes of at least one of the snapshots has no zero element, and $2p \leq m - 1$.

B. Co-Prime Array

Consider an $m = m_1 + 2m_2 - 1$ element co-prime array, consisting of two uniform subarrays: one with m_1 elements at locations $\mathbb{l}_1 = \{0, m_2, 2m_2, \dots, (m_1 - 1)m_2\}$ and the other with $2m_2 - 1$ elements at locations $\mathbb{l}_2 = \{m_1, 2m_1, \dots, (2m_2 - 1)m_1\}$, where $(m_1, m_2) = 1$ and $m_1 > m_2$. In this case, the generalized Vandermonde matrix $\mathbf{V}(\mathbf{z}, \mathbb{l}) \in \mathbb{C}^{m \times p}$ of modes may be partitioned as

$$\mathbf{V}(\mathbf{z}, \mathbb{l}) = \begin{bmatrix} \mathbf{V}(\mathbf{z}, \mathbb{l}_1) \\ \mathbf{V}(\mathbf{z}, \mathbb{l}_2) \end{bmatrix}, \quad (23)$$

where

$$\mathbf{V}(\mathbf{z}, \mathbb{l}_1) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1^{m_2} & z_2^{m_2} & \dots & z_p^{m_2} \\ z_1^{2m_2} & z_2^{2m_2} & \dots & z_p^{2m_2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(m_1-1)m_2} & z_2^{(m_1-1)m_2} & \dots & z_p^{(m_1-1)m_2} \end{bmatrix} \quad (24)$$

and

$$\mathbf{V}(\mathbf{z}, \mathbb{l}_2) = \begin{bmatrix} z_1^{m_1} & z_2^{m_1} & \dots & z_p^{m_1} \\ z_1^{2m_1} & z_2^{2m_1} & \dots & z_p^{2m_1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(2m_2-1)m_1} & z_2^{(2m_2-1)m_1} & \dots & z_p^{(2m_2-1)m_1} \end{bmatrix} \quad (25)$$

are the Vandermonde matrices for the two individual subarrays of the co-prime array. We assume the p modes $\{z_k\}_{k=1}^p$ are such that $z_i^{m_1 m_2} \neq z_j^{m_1 m_2}$ for $i \neq j$. This assumption is necessary for the identifiability of the modes from our approximate least squares algorithm, as will be shown in this section.

Let $\mathbf{A}_1 \in \mathbb{C}^{m_1 \times (m_1 - p)}$ and $\mathbf{B}_1 \in \mathbb{C}^{(2m_2 - 1) \times (2m_2 - 1 - p)}$ be matrices that are orthogonal to $\mathbf{V}(\mathbf{z}, \mathbb{l}_1)$ and $\mathbf{V}(\mathbf{z}, \mathbb{l}_2)$, respectively. That is, $\mathbf{A}_1^H \mathbf{V}(\mathbf{z}, \mathbb{l}_1) = \mathbf{0}$ and $\mathbf{B}_1^H \mathbf{V}(\mathbf{z}, \mathbb{l}_2) = \mathbf{0}$. Following our results in the restricted co-prime case, we may parameterize $\mathbf{A}_1 \in \mathbb{C}^{m_1 \times (m_1 - p)}$ as

$$\mathbf{A}_1 = \begin{bmatrix} a_p & a_{p-1} & \dots & a_1 & 1 & \dots & 0 \\ 0 & a_p & \dots & & & \dots & 0 \\ \vdots & \ddots & \ddots & & & & \vdots \\ 0 & \dots & 0 & a_p & \dots & a_1 & 1 \end{bmatrix}^H, \quad (26)$$

where $\{a_i\}_{i=1}^p$ are the coefficients of a polynomial $A(z)$, whose roots are $w_k = z_k^{m_2}$, $k = 1, 2, \dots, p$. That is,

$$\begin{aligned} A(z) &= \prod_{k=1}^p (1 - w_k z^{-1}) \\ &= \sum_{i=0}^p a_i z^{-i}, \quad a_0 = 1. \end{aligned} \quad (27)$$

Similarly, we parameterize $\mathbf{B}_1 \in \mathbb{C}^{(2m_2 - 1) \times (2m_2 - 1 - p)}$ as

$$\mathbf{B}_1 = \begin{bmatrix} b_p & b_{p-1} & \dots & b_1 & 1 & \dots & 0 \\ 0 & b_p & \dots & & & \dots & 0 \\ \vdots & \ddots & \ddots & & & & \vdots \\ 0 & \dots & 0 & b_p & \dots & b_1 & 1 \end{bmatrix}^H, \quad (28)$$

where $\{b_i\}_{i=1}^p$ are the coefficients of a polynomial $B(z)$, whose roots are $s_k = z_k^{m_1}$, $k = 1, 2, \dots, p$. That is,

$$\begin{aligned} B(z) &= \prod_{k=1}^p (1 - s_k z^{-1}) \\ &= \sum_{i=0}^p b_i z^{-i}, \quad b_0 = 1. \end{aligned} \quad (29)$$

Note that we still need p more independent columns to fully characterize the basis matrix \mathbf{A} for the orthogonal subspace $\langle \mathbf{A} \rangle$. However, using our partial characterization, we can estimate the modes (with no aliasing ambiguities) in the steps summarized in Algorithm 2.

Algorithm 2: Modal Estimation from the Co-Prime Array

- 1: Separate the measurements of the two subarrays as $\mathbf{u}[n] = \{y_i[n] | i \in \mathbb{l}_1\}$ and $\mathbf{v}[n] = \{y_i[n] | i \in \mathbb{l}_2\}$;
- 2: Estimate $\hat{\mathbf{a}} = [\hat{a}_1, \hat{a}_2, \dots, \hat{a}_p]^T$ from the following least squares problem for $\mathbf{u}[n]$:

$$\hat{\mathbf{a}} = \underset{\mathbf{a}}{\operatorname{argmin}} \sum_{n=0}^{N-1} \mathbf{u}^H[n] \mathbf{P}_{\mathbf{A}_1} \mathbf{u}[n],$$

where \mathbf{A}_1 is given in (26);

- 3: Root $\hat{A}(z) = 1 + \sum_{i=1}^p \hat{a}_i z^{-i}$ to return the roots $\{\hat{w}_k\}_{k=1}^p$. Then, recognizing that the m_2 th roots of \hat{w}_k are $\hat{z}_k e^{j2\pi q/m_2}$ for some $q \in \{0, 1, \dots, m_2 - 1\}$, form the set of candidate modes \mathcal{R}_1 as

$$\mathcal{R}_1 = \left\{ \left(\hat{z}_1 e^{j2\pi k_1/m_2}, \hat{z}_2 e^{j2\pi k_2/m_2}, \dots, \hat{z}_p e^{j2\pi k_p/m_2} \right) \mid 0 \leq k_1, k_2, \dots, k_p \leq m_2 - 1 \right\};$$

- 4: Estimate $\hat{\mathbf{b}} = [\hat{b}_1, \hat{b}_2, \dots, \hat{b}_p]^T$ from the following least squares problem for $\mathbf{v}[n]$:

$$\hat{\mathbf{b}} = \underset{\mathbf{b}}{\operatorname{argmin}} \sum_{n=0}^{N-1} \mathbf{v}^H[n] \mathbf{P}_{\mathbf{B}_1} \mathbf{v}[n],$$

where \mathbf{B}_1 is given in (28);

- 5: Root $\hat{B}(z) = 1 + \sum_{i=1}^p \hat{b}_i z^{-i}$ to return the roots $\{\hat{s}_k\}_{k=1}^p$. Then, recognizing that the m_1 th roots of \hat{s}_k are $\hat{z}_k e^{j2\pi q/m_1}$ for some $q \in \{0, 1, \dots, m_1 - 1\}$, form the set of candidate modes \mathcal{R}_2 as

$$\mathcal{R}_2 = \left\{ \left(\hat{z}_1 e^{j2\pi k_1/m_1}, \hat{z}_2 e^{j2\pi k_2/m_1}, \dots, \hat{z}_p e^{j2\pi k_p/m_1} \right) \mid 0 \leq k_1, k_2, \dots, k_p \leq m_1 - 1 \right\};$$

- 6: Intersect \mathcal{R}_1 and \mathcal{R}_2 , in other words look for the closest (based on the Euclidean metric) p members of the set \mathcal{R}_1 to the set \mathcal{R}_2 .
-

Theorem 3: Let $\mathbf{S}_{xx} \succ \mathbf{0}$, and $\mathbf{e}[n] = \mathbf{0}$, for $n = 0, 1, \dots, N - 1$, in (2). Consider a co-prime array with the set of sensor locations defined as $\mathbb{L} = \{0, m_2, 2m_2, \dots, (m_1 - 1)m_2\} \cup \{m_1, 2m_1, \dots, (2m_2 - 1)m_1\}$ for $(m_1, m_2) = 1$. Let there be p modes $\{z_k\}_{k=1}^p$, such that $z_i^{m_1 m_2} \neq z_j^{m_1 m_2}$ for $i \neq j$, and $p \leq \min\{m_1 - 1, 2m_2 - 2\}$. Then, Algorithm 2 uniquely identifies these modes from the co-prime array specified by \mathbb{L} .

Proof: See Appendix C.

Remark 6: Similar to Theorems 1 and 2, Theorem 3 requires $p \leq \min\{m_1 - 1, 2m_2 - 2, N\}$. However, the proof may be modified to hold for a single snapshot $\mathbf{y}[0] = \mathbf{V}(\mathbf{z}, \mathbb{L})\mathbf{x}[0]$, provided that no element of $\mathbf{x}[0]$ is zero, meaning no coordinate of the subspace $\langle \mathbf{V}(\mathbf{z}, \mathbb{L}) \rangle$ is excluded in the construction of $\mathbf{y}[0]$. In the single snapshot case, p modes $\{z_k\}_{k=1}^p$ ($z_i^{m_1 m_2} \neq z_j^{m_1 m_2}$ for $i \neq j$) can be identified provided that $2p \leq \min\{m_1, 2m_2 - 1\}$. The proof follows from a straightforward modification of the proof in Appendix C. Naturally, Theorem 3 still holds for $1 < N < p$ snapshots, if the mode amplitudes of at least one of the snapshots has no zero element, and $2p \leq \min\{m_1, 2m_2 - 1\}$.

Remark 7: Algorithm 2 solves two least squares problems in steps 2 and 4, to estimate the modes $\{z_k\}_{k=1}^p$. These least squares problems are constructed, using our parameterizations of the orthogonal subspaces in (26) and (28). Such parameterizations allow us to use subspace methods of modal analysis to efficiently solve these least squares problems and estimate the modes. In our numerical demonstrations in Section V.B, we use the IQML method (see, e.g., [7], [8], and [22]) in solving the least squares problems, but other subspace methods can also be used. In the noise free case, IQML iterations solve the least squares problems exactly. In fact, if in the IQML iterations $\mathbf{A}\mathbf{A}^H$ is initiated to an identity matrix, the algorithm returns the exact solutions in one iteration (see [7]).

Remark 8: To complete the $2p$ -parameterization of the basis matrix \mathbf{A} for the orthogonal subspace $\langle \mathbf{A} \rangle$, let $\mathbb{L}_{p_1} = \{0, m_2, \dots, (p-1)m_2\}$ and $\mathbb{L}_{p_2} = \{m_1, 2m_1, \dots, pm_1\}$. Then, the p remaining columns of \mathbf{A} may be represented in $\mathbf{C}_1 \in \mathbb{C}^{m \times p}$ as

$$\mathbf{C}_1^H = [\mathbf{C}_0^H | \mathbf{0}_{p \times (m_1 - p)} | \mathbf{I}_p | \mathbf{0}_{p \times (2m_2 - 1 - p)}] \quad (30)$$

where $\mathbf{0}_{k \times l}$ denotes a $k \times l$ matrix with zero entries, \mathbf{I}_p is the $p \times p$ identity matrix, and

$$\mathbf{C}_0^H = -\mathbf{V}(\mathbf{z}, \mathbb{L}_{p_2})\mathbf{V}^{-1}(\mathbf{z}, \mathbb{L}_{p_1}) \in \mathbb{C}^{p \times p}. \quad (31)$$

From (30) and (31) we can see that \mathbf{C}_1 only depends on $\{z_k^{m_1}\}_{k=1}^p$ and $\{z_k^{m_2}\}_{k=1}^p$ which are obtained from \mathbf{a} and \mathbf{b} by rooting $A(z)$ and $B(z)$ in (27) and (29), respectively. Therefore, the full, $2p$ -parameterized, characterization of the orthogonal subspace for the co-prime array may be written as

$$\mathbf{A} = \left[\begin{array}{cc|c} \mathbf{A}_1 & \mathbf{0} & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{B}_1 & \end{array} \right]. \quad (32)$$

We note that we do not need this full characterization for estimating the modes. The partial characterization using \mathbf{A}_1 and \mathbf{B}_1 suffices, at the expense of an extra p fitting equations.

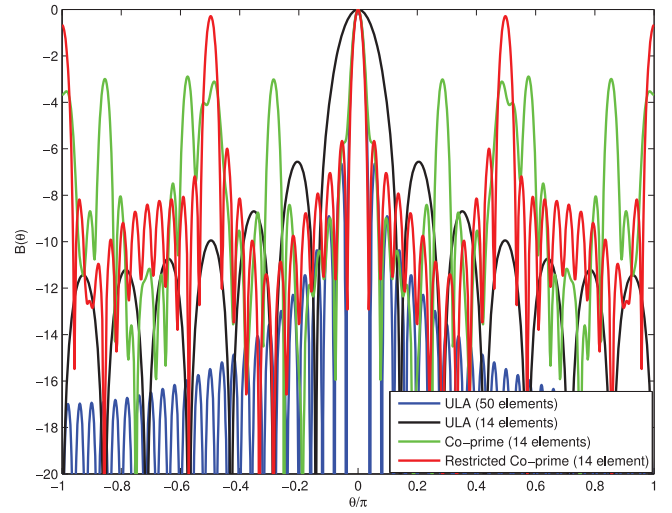


Fig. 2. Beam patterns for ULAs with 14 and 50 elements, a restricted co-prime array with 14 elements, $d = 4$, and $M = 3$, and a co-prime array with 14 elements, $m_1 = 7$, and $m_2 = 4$.

V. NUMERICAL RESULTS

In this section, we present numerical results for the estimation of damped and undamped complex exponential modes in co-prime and uniform line arrays, with the same (or almost the same) total apertures. We consider a ULA of 50 elements. We form our co-prime arrays with 14 elements by subsampling this ULA. For the restricted array, we subsample the measurements of the ULA by a factor of $d = 4$ and place a sensor at $M = 3$. For the co-prime array, the first subarray includes $m_1 = 7$ elements with interelement spacing of $m_2 = 4$, and the second subarray includes $2m_2 - 1 = 7$ elements with interelement spacing of $m_1 = 7$.

A. Beam pattern

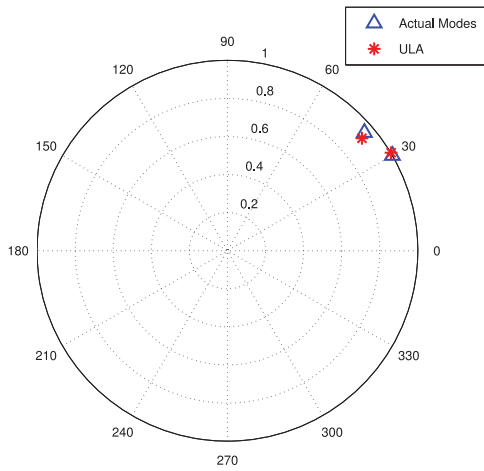
It is insightful to first look at the beam patterns of restricted co-prime, co-prime, and uniform line arrays for the problem of estimating undamped modes. In this case, the beam pattern $B(\theta)$ is

$$B(\theta) = \sum_{l=0}^{m-1} e^{jil\theta}. \quad (33)$$

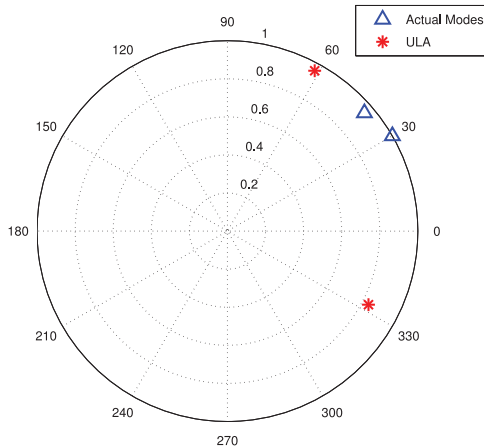
Fig. 2 shows the beam patterns for different array geometries. Although the co-prime arrays of 14 elements have the same aperture and the same main lobe width as the 50-element ULA, we see that they have higher sidelobes, suggesting that there will be performance losses in resolving closely spaced modes using these arrays, relative to the ULA, especially if noise fields do not produce white noise at the sensor array.

B. Numerical Demonstrations

As demonstrations of Algorithm 1 and Algorithm 2, we offer Figs. 3–5. These results are generated using IQML (see [7], [8], and [22]) to solve the least squares problems in step 1 of Algorithm 1 and steps 2 and 4 of Algorithm 2, but any other subspace method would produce similar results at these SNRs. The two



(a)



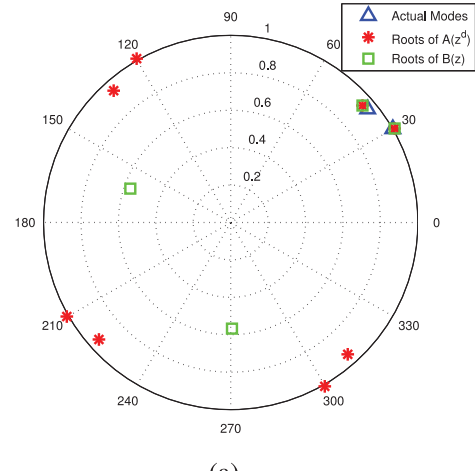
(b)

Fig. 3. ULA. Estimating two closely spaced modes $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ using a ULA with 50 elements: (a) Per sensor SNR = 0 dB, (b) Per sensor SNR = -5 dB.

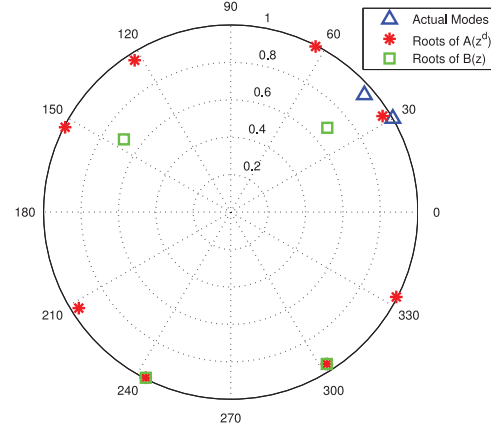
modes to be estimated here are $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$. Fig. 3 illustrates mode identification at two SNRs, one high, where the modes are identified, and one low, where they are misidentified. These results are illustrative with no claims about estimation accuracy for randomly generated data. Figs. 4 and 5 are the same demonstrations for the restricted co-prime and co-prime arrays, at SNRs 5 dB higher than for the ULA. The three extra mode pairs, eventually resolved in the left-hand pictures, are the ambiguous modes introduced by subsampling. In all of these examples, we are given one snapshot at the specified SNR.

C. Fisher Information, CRB, and Mean-Squared Error

Let us also look at the Fisher information matrix for the co-prime and uniform line arrays. For the proper complex Gaussian measurement model in (2), let $\boldsymbol{\theta} =$



(a)



(b)

Fig. 4. Restricted co-prime. Estimating two closely spaced modes $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ using a restricted co-prime array with 14 elements, $d = 4$ and $M = 3$: (a) Per sensor SNR = 5 dB, (b) Per sensor SNR = 0 dB.

$[\mathbf{z}^T, \mathbf{x}^T[0], \dots, \mathbf{x}^T[N-1]]^T$ be the set of complex parameters. The Fisher information matrix for the parameter vector $\boldsymbol{\theta}$ is [29]

$$\mathbf{J}_F(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}} \left\{ \left[\frac{\partial}{\partial \boldsymbol{\theta}} \log \left(\prod_{n=0}^{N-1} p_{\boldsymbol{\theta}}(\mathbf{y}[n]) \right) \right]^H \times \frac{\partial}{\partial \boldsymbol{\theta}} \log \left(\prod_{n=0}^{N-1} p_{\boldsymbol{\theta}}(\mathbf{y}[n]) \right) \right\}, \quad (34)$$

where

$$p_{\boldsymbol{\theta}}(\mathbf{y}[n]) = \frac{1}{(\pi\sigma^2)^m} \times \exp \left\{ -\frac{1}{\sigma^2} (\mathbf{y}[n] - \mathbf{V}(\mathbf{z})\mathbf{x}[n])^H (\mathbf{y}[n] - \mathbf{V}(\mathbf{z})\mathbf{x}[n]) \right\}. \quad (35)$$

The CRB matrix for the estimation of the p complex modes $\mathbf{z} = [z_1, \dots, z_p]^T$ from the data model in (2) and (3) is

$$\mathbf{CRB}(\mathbf{z}, \mathbb{I}) = \sigma^2 \left[\sum_{n=0}^{N-1} \mathbf{X}^H[n] \mathbf{D}^H(\mathbf{z}, \mathbb{I}) [\mathbf{I} - \mathbf{P}_{\mathbf{V}(\mathbf{z}, \mathbb{I})}] \mathbf{D}(\mathbf{z}, \mathbb{I}) \mathbf{X}[n] \right]^{-1} \quad (36)$$

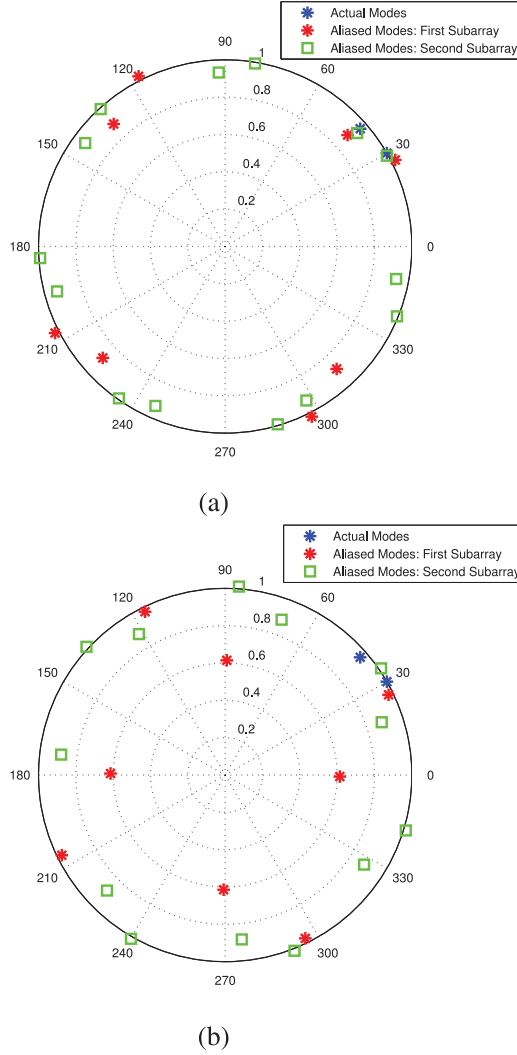


Fig. 5. Co-prime. Estimating two closely spaced modes $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ using a co-prime array with 14 elements, $m_1 = 7$ and $m_2 = 4$: (a) Per sensor SNR = 5 dB, (b) Per sensor SNR = 0 dB.

where $\mathbf{X}[n] = \text{diag}(x_1[n], \dots, x_p[n])$, $\mathbb{1}$ is the set of sensor locations, and

$$\mathbf{D}(\mathbf{z}, \mathbb{1}) = \begin{bmatrix} i_0 z_1^{i_0-1} & i_0 z_2^{i_0-1} & \dots & i_0 z_p^{i_0-1} \\ i_1 z_1^{i_1-1} & i_1 z_2^{i_1-1} & \dots & i_1 z_p^{i_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ i_{m-1} z_1^{i_{m-1}-1} & i_{m-1} z_2^{i_{m-1}-1} & \dots & i_{m-1} z_p^{i_{m-1}-1} \end{bmatrix}. \quad (37)$$

The derivation of (36) is given in Appendix D. It follows the methodology of [28] and [30], using the complex calculus of [29].

Remark 9: Relative to the CRB in the single snapshot case, the CRB in (36) decreases like $\frac{1}{N}$ in the number N of snapshots.

Fig. 6 shows the concentration ellipses (see, e.g., [27], [30]) for the Fisher information matrix for the estimation of two correlated modes at $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ for the ULA, co-prime and restricted co-prime arrays, at three SNRs with N

= 10 snapshots. Here, we have assumed that the experimental correlation matrix of the two sources is

$$\sum_{n=0}^{N-1} \mathbf{x}[n] \mathbf{x}^H[n] / N = \sigma_x^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad (38)$$

with $\rho = 0.8$. This choice of ρ is arbitrary, but is used to illustrate the point that the CRB does not require any special averaging conditions for \mathbf{S}_{xx} . The ellipses of Fig. 6 are the loci of all points $\mathbf{e} = [e_1 e_2]^T$ for which $\mathbf{e}^H [\mathbf{CRB}(\mathbf{z}, \mathbb{1})]^{-1} \mathbf{e} = c$, for the CRB matrix $\mathbf{CRB}(\mathbf{z}, \mathbb{1})$ in (36), and for a constant level c . To obtain two dimensional ellipses we assume the complex errors $e_1 = \hat{z}_1 - z_1$ and $e_2 = \hat{z}_2 - z_2$ are parameterized as $e_1 = a_1 + j b_1$ and $e_2 = a_2 + j b_2$. In order to get viewable concentration ellipses we slice through the 4-dimensional ellipsoid to get 2-dimensional ellipses. For example, with $b_1 = b_2 = 0$ we get the concentration slice for (a_1, a_2) . Fig. 6 is this case, but all other slices are similar. As Fig. 6 shows, the concentration ellipses for the restricted and conventional co-prime arrays are similar, but they are larger than the concentration ellipse for the ULA. The concentration ellipses for the co-prime arrays at SNR of 0 dB are essentially the same as the concentration ellipses for the ULA at SNR of -5 dB, suggesting the same error variances at these SNRs.

Fig. 7 shows the mean-squared error (MSE) plots versus SNR (averaged over 1000 trials) in the estimation of two modes at $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$ and for the ULA with 50 elements and the restricted co-prime array with 14 elements $d = 4$ and $M = 3$, and for $N = 10$ snapshots. The amplitudes of the two modes are correlated with correlation factor $\rho = 0.8$. These results are generated using reduced rank linear prediction (see [9] and [10]) in solving the least squares problem in (6) for the dense array, and the least squares problem in step 1 of Algorithm 1 for the restricted co-prime array. Fig. 7(a) shows the total MSEs and the CRBs for estimating the two modes, and Fig. 7(b) shows the MSEs and the CRBs in estimating the damped mode $z_2 = 0.95e^{j0.69}$, in the presence of the interfering mode $z_1 = e^{j0.52}$. The co-prime array with $m_1 = 7$ and $m_2 = 4$ has the same number of elements and almost the same aperture as the restricted co-prime array in this example, and therefore would produce similar MSE and CRB curves.

These results demonstrate a loss of about $10 \log_{10} 50/14$ dB (at high SNR) in estimation accuracy with co-prime arrays relative to the ULA. Since the apertures of the co-prime arrays are equal to the aperture of the dense array, the smaller number of sensors in the co-prime arrays accounts for the inflation of the concentration ellipses in Fig. 6, and the increase in MSEs in Fig. 7. This finding does not conflict with the findings of [3], where the authors studied CRBs for ULA and co-prime arrays which had the same number of elements. In that case, the co-prime array had a larger aperture and consequently enjoyed resolution gains over the ULA when sources are undamped and temporal samples are rich enough for $\rho = 0$ in (38).

VI. CONCLUSION

We have considered the problem of estimating the parameters of complex exponential modes, directly from co-prime sam-

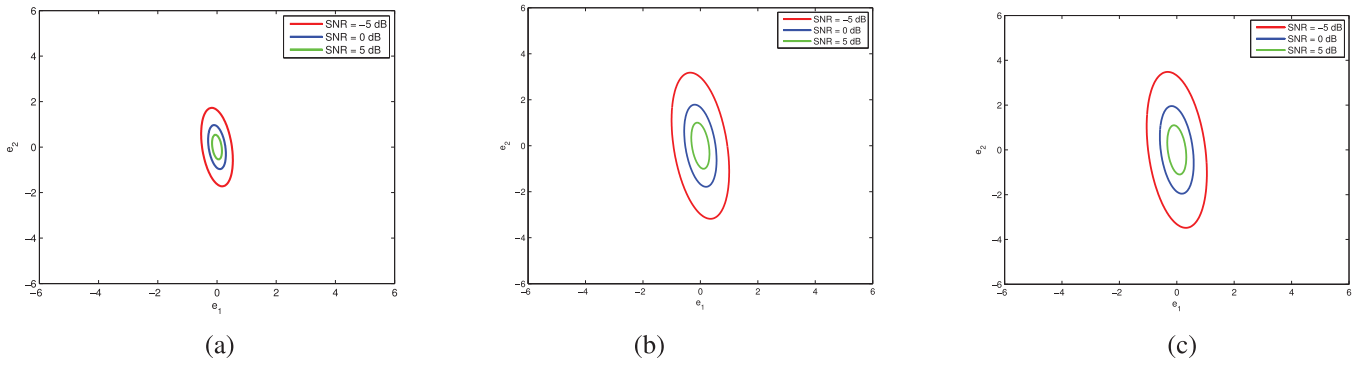


Fig. 6. Concentration ellipses for the Fisher information matrix on the estimation of two modes at $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$: (a) ULA with 50 elements. (b) restricted co-prime array with 14 elements $d = 4$ and $M = 3$. (c) co-prime array with 14 elements $m_1 = 7$ and $m_2 = 4$. (a) ULA, (b) Restricted co-prime array, (c) Co-prime array.

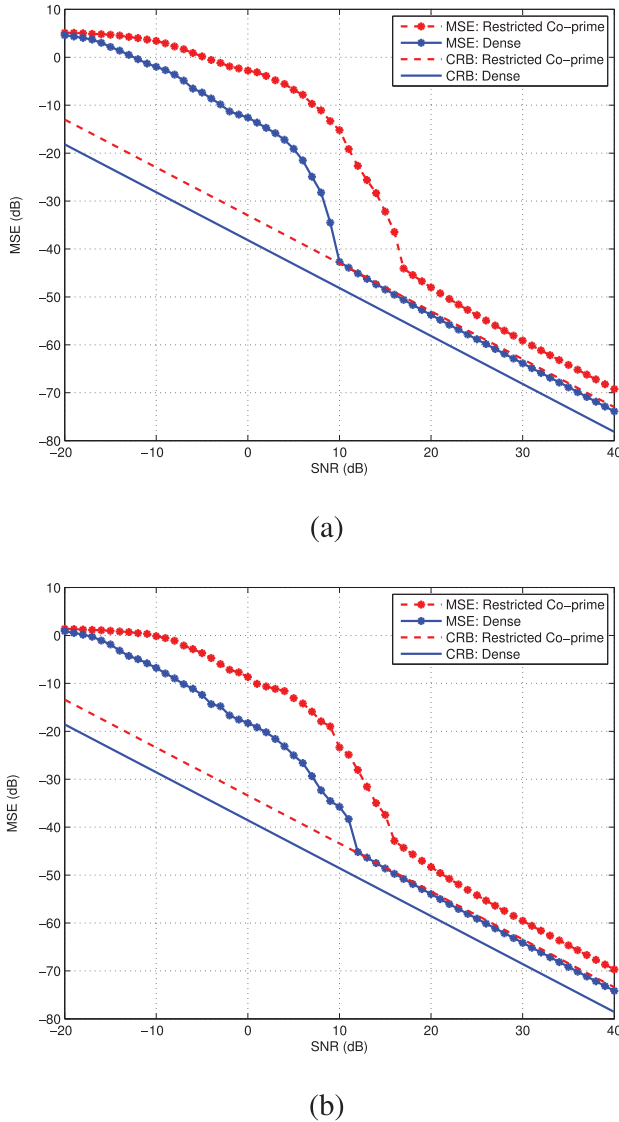


Fig. 7. (a) Total MSEs and CRBs in estimating $z_1 = e^{j0.52}$ and $z_2 = 0.95e^{j0.69}$. (b) MSEs and CRBs in estimating $z_2 = 0.95e^{j0.69}$ in the presence of an interfering mode $z_1 = e^{j0.52}$. The ULA and the restricted co-prime array have equal apertures. The ULA has 50 elements and the restricted co-prime array has 14 elements, with $d = 4$, and $M = 3$.

ples (in time or space) of their weighted sum. The modes can be damped or undamped and snapshots are not plentiful. We have

derived *identifiability* theorems and algorithms, and have determined parameterizations of a subspace that is orthogonal to the signal subspace spanned by the columns of the generalized Vandermonde matrix of the modes in a co-prime array. Our parameterizations are of a form that is particularly suitable for utilizing methods based on subspace signal processing (see [7]–[12], and [22]), for estimating the modes.

For performance analysis, we have studied CRBs, error concentration ellipses, and mean-squared errors (MSEs) in the case where the aperture of a co-prime array equals the aperture of a ULA, but many fewer sensors are used for the co-prime arrays than would be used in the ULA. This is the typical case in radar and sonar, where the limiting factors are aperture and observation time. We have also presented numerical examples demonstrating the way the proposed modal estimation method resolves aliases. The CRB results bound MSE *performance* at various SNRs for ULAs and co-prime samplings of them. These results quantify the loss in performance due to the use of fewer sensors in a co-prime array.

APPENDIX A PROOF OF THEOREM 1

Let $\mathbf{y}[n] = \mathbf{V}(\mathbf{z}, \mathbb{l})\mathbf{x}[n]$ for $n = 0, \dots, N - 1$, where $\mathbf{V}(\mathbf{z}, \mathbb{l}) = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_p, \mathbb{l})]$ is defined in (3) for any arbitrary distinct modes $\mathbf{z} = [z_1, \dots, z_p]^T$, and a given set of sensor location indices \mathbb{l} . Let $\mathbf{S}_{xx} = \sum_{n=0}^{N-1} \mathbf{x}[n]\mathbf{x}^H[n]/N$ be the rank- p correlation matrix of the sources, $N \geq p$.

IF. Assume $\mathbf{V}_{aug} = [\mathbf{v}(u_1), \dots, \mathbf{v}(u_{p+1})]$ is full rank $p + 1$ for any set of $p + 1$ distinct modes $\{u_k\}_{k=1}^{p+1}$. There exists a full rank matrix $\mathbf{A}(\mathbf{z}, \mathbb{l}) \in \mathbb{C}^{m \times (m-p)}$, such that $\mathbf{A}^H(\mathbf{z}, \mathbb{l})\mathbf{V}(\mathbf{z}, \mathbb{l}) = \mathbf{0}$, and the dimension of the null space of $\mathbf{A}^H(\mathbf{z}, \mathbb{l})$ is p . Then, we have

$$\sum_{n=0}^{N-1} \mathbf{y}^H[n] \mathbf{P}_{\mathbf{A}(\mathbf{z}, \mathbb{l})} \mathbf{y}[n] = N \text{tr} [\mathbf{P}_{\mathbf{A}(\mathbf{z}, \mathbb{l})} \mathbf{V}(\mathbf{z}, \mathbb{l}) \mathbf{S}_{xx} \mathbf{V}^H(\mathbf{z}, \mathbb{l})] = 0. \quad (39)$$

Now, assume $\mathbf{w} = [w_1, \dots, w_p]^T$ ($w_i \neq w_j$ for $i \neq j$) is another solution to (6), that is

$$\text{tr} [\mathbf{P}_{\mathbf{A}(\mathbf{w}, \mathbb{l})} \mathbf{V}(\mathbf{z}, \mathbb{l}) \mathbf{S}_{xx} \mathbf{V}^H(\mathbf{z}, \mathbb{l})] = 0, \quad (40)$$

where $\mathbf{A}(\mathbf{w}, \mathbb{l}) \in \mathbb{C}^{m \times (m-p)}$ is a full column rank matrix such that $\mathbf{A}^H(\mathbf{w}, \mathbb{l})\mathbf{V}(\mathbf{w}, \mathbb{l}) = \mathbf{0}$, and the dimension of the null space of $\mathbf{A}^H(\mathbf{w}, \mathbb{l})$ is p . From (40), we have

$$\mathbf{P}_{\mathbf{A}(\mathbf{w}, \mathbb{l})}\mathbf{V}(\mathbf{z}, \mathbb{l})\mathbf{S}_{xx}^{1/2} = \mathbf{0}. \quad (41)$$

Now, because \mathbf{S}_{xx} is full rank, (41) implies that

$$\langle \mathbf{V}(\mathbf{z}, \mathbb{l}) \rangle \subseteq \text{Null}(\mathbf{A}^H(\mathbf{w}, \mathbb{l})) = \langle \mathbf{V}(\mathbf{w}, \mathbb{l}) \rangle. \quad (42)$$

From (42), the rank of $[\mathbf{v}(w_1), \dots, \mathbf{v}(w_p), \mathbf{v}(z_k)]$ is p , for $k = 1, \dots, p$, which is only possible if $z_k \in \{w_1, \dots, w_p\}$, for $k = 1, \dots, p$. Therefore, $\mathbf{w} = \mathbf{z}$, which means $\{z_k\}_{k=1}^p$ is the unique solution to (6).

Only If. For the converse, assume there exists $w \notin \{z_k\}_{k=1}^p$ such that the rank of

$$\mathbf{V}_{aug} = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_p, \mathbb{l}), \mathbf{v}(w, \mathbb{l})] \quad (43)$$

is less than $p + 1$. Because \mathbf{V}_{aug} in (43) is rank deficient, there exists a $z_i, 1 \leq i \leq p$ such that $\mathbf{v}(z_i) \in \langle [\mathbf{v}(z_1), \dots, \mathbf{v}(z_{i-1}), \mathbf{v}(z_{i+1}), \dots, \mathbf{v}(w)] \rangle$. Without loss of generality, assume

$$\mathbf{v}(z_p) \in \langle [\mathbf{v}(z_1), \dots, \mathbf{v}(z_{p-1}), \mathbf{v}(w)] \rangle. \quad (44)$$

Let $\mathbf{w} = [z_1, \dots, z_{p-1}, w]^T$, and define $\mathbf{A}(\mathbf{w}, \mathbb{l}) \in \mathbb{C}^{m \times (m-p)}$ as a full column rank matrix such that $\mathbf{A}^H(\mathbf{w}, \mathbb{l})\mathbf{V}(\mathbf{w}, \mathbb{l}) = \mathbf{0}$, and the dimension of the null space of $\mathbf{A}^H(\mathbf{w}, \mathbb{l})$ is p . Then, from (44) we have

$$\mathbf{A}^H(\mathbf{w}, \mathbb{l})\mathbf{V}(\mathbf{z}, \mathbb{l}) = \mathbf{0}. \quad (45)$$

Therefore,

$$\sum_{n=0}^{N-1} \mathbf{y}^H[n]\mathbf{P}_{\mathbf{A}(\mathbf{z}, \mathbb{l})}\mathbf{y}[n] = \sum_{n=0}^{N-1} \mathbf{y}^H[n]\mathbf{P}_{\mathbf{A}(\mathbf{w}, \mathbb{l})}\mathbf{y}[n] = 0, \quad (46)$$

which implies that \mathbf{z} and \mathbf{w} both minimize (6). Thus, there is no unique solution to (6). ■

Remark 10: A sketch of the proof for the single snapshot case goes like this: Let $\mathbf{y}[0] = \mathbf{V}(\mathbf{z}, \mathbb{l})\mathbf{x}[0]$, where $\mathbf{V}(\mathbf{z}, \mathbb{l}) = [\mathbf{v}(z_1, \mathbb{l}), \dots, \mathbf{v}(z_p, \mathbb{l})]$ is defined in (3) for any arbitrary distinct modes $\mathbf{z} = [z_1, \dots, z_p]^T$, and $\mathbf{x}[0]$ has no zero element. There exists a unique solution for (6) if and only if

$$\mathbf{y}^H[0]\mathbf{P}_{\mathbf{A}(\mathbf{w}, \mathbb{l})}\mathbf{y}[0] \neq 0 \quad \text{for } \mathbf{w} \neq \mathbf{z}, \quad (47)$$

where the elements of \mathbf{w} are distinct. This is equivalent to

$$\mathbf{A}^H(\mathbf{w}, \mathbb{l})\mathbf{V}(\mathbf{z}, \mathbb{l})\mathbf{x}[0] \neq 0 \quad \text{for } \mathbf{w} \neq \mathbf{z}, \quad (48)$$

for any $\mathbf{x}[0]$ with no zero element. But (48) holds if and only if $\mathbf{V}(\{z_k\}_{k=1}^p \cup \{w_k\}_{k=1}^p)$ is full rank for all arbitrary choices of distinct $\{w_k\}_{k=1}^p$ and distinct $\{z_k\}_{k=1}^p$. This happens if and only if the $m \times 2p$ matrix $V_{aug}(z_1, \dots, z_{2p})$ is full rank for any $2p$ distinct modes $\{z_k\}_{k=1}^{2p}$.

APPENDIX B

PROOF OF THEOREM 2

Let $p \leq m - 2$, and $\mathbf{z} = [z_1, \dots, z_p]^T$ be the vector of p modes such that $z_i^d \neq z_j^d$ for $i \neq j$. Also, let $\mathbf{a} = (a_1, a_2, \dots, a_p)^T$ be the solution to (19). That is,

$$\begin{aligned} \mathbf{a} &= \underset{\boldsymbol{\alpha}}{\text{argmin}} \sum_{n=0}^{N-1} \mathbf{y}^H[n]\mathbf{P}_{\mathbf{A}_0(\boldsymbol{\alpha})}\mathbf{y}[n] \\ &= \underset{\boldsymbol{\alpha}}{\text{argmin}} \text{tr} [\mathbf{P}_{\mathbf{A}_0(\boldsymbol{\alpha})}\mathbf{V}(\mathbf{z}, \mathbb{l}_s)\mathbf{S}_{xx}\mathbf{V}^H(\mathbf{z}, \mathbb{l}_s)], \end{aligned} \quad (49)$$

where $\mathbf{A}_0(\boldsymbol{\alpha})$ denotes an \mathbf{A}_0 of the form (13), with α_i 's replacing a_i 's for $i = 1, 2, \dots, p$, $\mathbf{S}_{xx} = \sum_{n=0}^{N-1} \mathbf{x}[n]\mathbf{x}^H[n]/N \succ \mathbf{0}$, and $\mathbf{V}(\mathbf{z}, \mathbb{l}_s)$ is given by (11).

We first show that in the noiseless case the roots of the polynomial $A(z) = 1 + \sum_{i=1}^p a_i z^{-i}$ are $w_k = z_k^d$ for $k = 1, 2, \dots, p$ (the d th power of the actual modes).

Defining $\mathbf{B}(\boldsymbol{\alpha}) = (\mathbf{A}_0^H(\boldsymbol{\alpha})\mathbf{A}_0(\boldsymbol{\alpha}))^{-1/2}\mathbf{A}_0^H(\boldsymbol{\alpha})\mathbf{V}(\mathbf{z}, \mathbb{l}_s)\mathbf{S}_{xx}^{1/2}$, the minimization problem in (49) can be written as

$$\mathbf{a} = \underset{\boldsymbol{\alpha}}{\text{argmin}} \text{tr} [\mathbf{B}^H(\boldsymbol{\alpha})\mathbf{B}(\boldsymbol{\alpha})]. \quad (50)$$

The minimum value of the objective function in (50) is zero, for $\mathbf{B}(\mathbf{a}) = \mathbf{0}$, and because \mathbf{S}_{xx} and $\mathbf{A}_0^H(\mathbf{a})\mathbf{A}_0(\mathbf{a})$ are full rank, for the solution vector \mathbf{a} , from $\mathbf{B}(\mathbf{a}) = \mathbf{0}$ we have

$$\mathbf{A}_0^H(\mathbf{a})\mathbf{V}(\mathbf{z}, \mathbb{l}_s) = \mathbf{0}, \quad (51)$$

which we can reorder to get

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1^d & z_2^d & \dots & z_p^d \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{(m-p-2)d} & z_2^{(m-p-2)d} & \dots & z_p^{(m-p-2)d} \end{bmatrix} \times \begin{bmatrix} w_1^p A(w_1) & 0 & \dots & 0 \\ 0 & w_2^p A(w_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_p^p A(w_p) \end{bmatrix} = \mathbf{0}. \quad (52)$$

But (52) holds if and only if

$$A(w_k) = 0 \quad \text{for } k = 1, \dots, p, \quad (53)$$

where $w_i \neq w_j$ for $i \neq j$.

Now, let $\boldsymbol{\beta}$ be the solution to (22). Here we use $\boldsymbol{\beta}$ instead of $\hat{\mathbf{b}}$ for the solution to distinguish noiseless and noisy cases. We show $\boldsymbol{\beta}$ solves (16) and therefore resolves aliasing.

Based on the argument resulting in (53), the set of roots \mathcal{R} may be written as

$$\mathcal{R} = \left\{ \left(z_1 e^{j2\pi k_1/d}, z_2 e^{j2\pi k_2/d}, \dots, z_p e^{j2\pi k_p/d} \right) \mid 0 \leq k_1, k_2, \dots, k_p \leq d-1 \right\}, \quad (54)$$

where $\{z_k\}_{k=1}^p$ are the actual modes. In this case (22) can be rewritten as

$$\begin{aligned} \boldsymbol{\beta} &= \underset{\boldsymbol{\zeta}}{\text{argmin}} \sum_{n=0}^{N-1} \left| y_{m-1}[n] + \boldsymbol{\zeta}^T \mathbf{u}[n] \right|^2 \\ \text{s.t. } \mathbf{V}_p^T(\boldsymbol{\eta})\boldsymbol{\zeta} &= -\boldsymbol{\eta}^{\odot M}, \quad \boldsymbol{\eta} \in \mathcal{R}, \end{aligned} \quad (55)$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_p)$, $\boldsymbol{\eta}^{\odot M} = [\eta_1^M, \eta_2^M, \dots, \eta_p^M]^T$ and

$$\mathbf{V}_p(\boldsymbol{\eta}) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \eta_1^d & \eta_2^d & \dots & \eta_p^d \\ \vdots & \vdots & \ddots & \vdots \\ \eta_1^{(p-1)d} & \eta_2^{(p-1)d} & \dots & \eta_p^{(p-1)d} \end{bmatrix}. \quad (56)$$

In the noiseless case, $\mathbf{u}[n] = \mathbf{V}_p(\mathbf{z})\mathbf{x}[n]$, $y_{m-1}[n] = \sum_{k=1}^p x_k[n]z_k^M$ and $\mathbf{V}_p(\boldsymbol{\eta}) = \mathbf{V}_p(\mathbf{z})$. Therefore, we have

$$\begin{aligned} & \sum_{n=0}^{N-1} \left| y_{m-1}[n] + \boldsymbol{\beta}^T \mathbf{u}[n] \right|^2 \\ &= \sum_{n=0}^{N-1} \left| \sum_{k=1}^p x_k[n]z_k^M + \boldsymbol{\beta}^T \mathbf{V}_p(\mathbf{z})\mathbf{x}[n] \right|^2 \\ &= \sum_{n=0}^{N-1} \left| \sum_{k=1}^p x_k[n](z_k^M - \eta_k^M) \right|^2 \\ &= N \left\| \mathbf{S}_{xx}^{T/2}(\mathbf{z}^{\odot M} - \boldsymbol{\eta}^{\odot M}) \right\|_2^2 \geq 0. \end{aligned} \quad (57)$$

Now, because $\boldsymbol{\beta}$ is the solution to (55), then in the noiseless case $\sum_{n=0}^{N-1} |y_{m-1}[n] + \boldsymbol{\beta}^T \mathbf{u}[n]|^2 = 0$ and because \mathbf{S}_{xx} is full rank, from (57) we have $\boldsymbol{\eta}^{\odot M} = \mathbf{z}^{\odot M}$. Therefore,

$$\begin{aligned} \boldsymbol{\beta} &= -(\mathbf{V}_p^T(\boldsymbol{\eta}))^{-1} \boldsymbol{\eta}^{\odot M} \\ &= -(\mathbf{V}_p^T(\mathbf{z}))^{-1} \mathbf{z}^{\odot M} \\ &= \mathbf{b}, \end{aligned} \quad (58)$$

where \mathbf{b} is the solution to (16). Therefore, the intersection of the roots of $B(z)$ and $A(z^d)$ are (z_1, z_2, \dots, z_p) which are the actual modes to be estimated. ■

APPENDIX C PROOF OF THEOREM 3

Let $p \leq \min\{m_1 - 1, 2m_2 - 2\}$, and $z_i^{m_1 m_2} \neq z_j^{m_1 m_2}$ for $i \neq j$ (which implies $z_i^{m_1} \neq z_j^{m_1}$ and $z_i^{m_2} \neq z_j^{m_2}$ for $i \neq j$). In the noiseless case, because $p \leq m_1 - 1$ and $z_i^{m_2} \neq z_j^{m_2}$ for $i \neq j$, similar to the first part of the proof for Theorem 2, steps 2-3 of Algorithm 2 provide us with the set of candidate modes \mathcal{R}_1 as

$$\mathcal{R}_1 = \left\{ \left(z_1 e^{j2\pi k_1/m_2}, z_2 e^{j2\pi k_2/m_2}, \dots, z_p e^{j2\pi k_p/m_2} \right) \mid 0 \leq k_1, k_2, \dots, k_p \leq m_2 - 1 \right\},$$

where $\{z_k\}_{k=1}^p$ are the actual modes. Also, because $p \leq 2m_2 - 2$ and $z_i^{m_1} \neq z_j^{m_1}$ for $i \neq j$, steps 4-5 result in the set of candidate modes \mathcal{R}_2 as

$$\mathcal{R}_2 = \left\{ \left(z_1 e^{j2\pi k_1/m_1}, z_2 e^{j2\pi k_2/m_1}, \dots, z_p e^{j2\pi k_p/m_1} \right) \mid 0 \leq k_1, k_2, \dots, k_p \leq m_1 - 1 \right\}.$$

Now, assume there exists a common mode between the two sets \mathcal{R}_1 and \mathcal{R}_2 , that is

$$z_r e^{j2\pi k_r/m_2} = z_q e^{j2\pi \rho_q/m_1} \quad (59)$$

for some $0 \leq r, q \leq p - 1$, $0 \leq k_r \leq m_2 - 1$, and $0 \leq \rho_q \leq m_1 - 1$. There are two possible cases for (59):

Case 1: $r = q$, thus $e^{j2\pi k_r/m_2} = e^{j2\pi \rho_r/m_1}$ and we have

$$\frac{k_r}{m_2} - \frac{\rho_r}{m_1} = \ell \quad \text{for } \ell \in \mathbb{Z}, \quad (60)$$

for some $0 \leq k_r \leq m_2 - 1$ and $0 \leq \rho_r \leq m_1 - 1$. Now, because $(m_2, m_2) = 1$, (60) is only possible for $k_r = \rho_r = 0$, which gives us the r -th actual mode z_r .

Case 2: $r \neq q$, in this case (59) is impossible to hold, because $z_r^{m_1 m_2} \neq z_q^{m_1 m_2}$ for $r \neq q$.

Therefore, the only common elements of \mathcal{R}_1 and \mathcal{R}_2 are $\{z_k\}_{k=1}^p$, which are the actual modes. ■

APPENDIX D

FISHER INFORMATION MATRIX AND THE CRB

The logarithm of the joint pdf of $\mathbf{y}[0], \dots, \mathbf{y}[N-1]$ in (35) is

$$\log L = \log \left(\prod_{n=0}^{N-1} p_{\boldsymbol{\theta}}(\mathbf{y}[n]) \right) = -Nm \log \pi \sigma^2 - \frac{N}{\sigma^2} \text{tr}(\mathbf{S}_{yy}), \quad (61)$$

where $\mathbf{S}_{yy} = N^{-1} \sum_{n=0}^{N-1} (\mathbf{y}[n] - \mathbf{V}(\mathbf{z}, \mathbb{1})\mathbf{x}[n])(\mathbf{y}[n] - \mathbf{V}(\mathbf{z}, \mathbb{1})\mathbf{x}[n])^H$. Taking the partial derivatives of (61) with respect to the complex parameters $\boldsymbol{\theta} = [\mathbf{z}^T, \mathbf{x}^T[0], \dots, \mathbf{x}^T[N-1]]^T$ we have,

$$\frac{\partial \log L}{\partial \mathbf{z}} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \mathbf{e}^H[n] \mathbf{D}(\mathbf{z}, \mathbb{1}) \mathbf{X}[n], \quad (62)$$

$$\frac{\partial \log L}{\partial \mathbf{x}[n]} = \frac{1}{\sigma^2} \mathbf{e}^H[n] \mathbf{V}(\mathbf{z}, \mathbb{1}) \quad \text{for } n = 0, \dots, N-1, \quad (63)$$

where $\mathbf{e}[n] = \mathbf{y}[n] - \mathbf{V}(\mathbf{z}, \mathbb{1})\mathbf{x}[n]$, as in (2). Using the fact that $E[\mathbf{e}[n]\mathbf{e}^H[s]] = \delta[n-s]\sigma^2\mathbf{I}$, where $\delta[\cdot]$ is the Kronecker delta function, we have

$$\begin{aligned} \mathbf{J}(\mathbf{z}, \mathbf{z}) &= E \left[\begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{z}} \\ \frac{\partial \log L}{\partial \mathbf{z}} \end{pmatrix} \begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{z}} \\ \frac{\partial \log L}{\partial \mathbf{z}} \end{pmatrix} \right] \\ &= \frac{1}{\sigma^4} E \left[\sum_{n=0}^{N-1} \sum_{s=0}^{N-1} \mathbf{X}^H[n] \right. \\ &\quad \left. \times \mathbf{D}^H(\mathbf{z}, \mathbb{1}) \mathbf{e}[n] \mathbf{e}^H[s] \mathbf{D}(\mathbf{z}, \mathbb{1}) \mathbf{X}[s] \right] \\ &= \frac{1}{\sigma^2} E \left[\sum_{n=0}^{N-1} \mathbf{X}^H[n] \mathbf{D}^H(\mathbf{z}, \mathbb{1}) \mathbf{D}(\mathbf{z}, \mathbb{1}) \mathbf{X}[n] \right], \end{aligned} \quad (64)$$

$$\begin{aligned} \mathbf{J}(\mathbf{x}[k], \mathbf{x}[s]) &= E \left[\begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{x}[k]} \\ \frac{\partial \log L}{\partial \mathbf{x}[k]} \end{pmatrix} \begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{x}[s]} \\ \frac{\partial \log L}{\partial \mathbf{x}[s]} \end{pmatrix} \right] \\ &= \frac{1}{\sigma^4} E \left[\mathbf{V}^H(\mathbf{z}, \mathbb{1}) \mathbf{e}[k] \mathbf{e}^H[s] \mathbf{V}(\mathbf{z}, \mathbb{1}) \right] \\ &= \frac{1}{\sigma^2} \mathbf{V}^H(\mathbf{z}, \mathbb{1}) \mathbf{V}(\mathbf{z}, \mathbb{1}) \delta[k-s] \end{aligned} \quad (65)$$

for $k, s = 0, \dots, N-1$, and

$$\begin{aligned} \mathbf{J}(\mathbf{z}, \mathbf{x}[k]) &= E \left[\begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{z}} \\ \frac{\partial \log L}{\partial \mathbf{z}} \end{pmatrix} \begin{pmatrix} \frac{\partial \log L}{\partial \mathbf{x}[k]} \\ \frac{\partial \log L}{\partial \mathbf{x}[k]} \end{pmatrix} \right] \\ &= \frac{1}{\sigma^4} E \left[\sum_{n=0}^{N-1} \mathbf{X}^H[n] \mathbf{D}^H(\mathbf{z}, \mathbb{1}) \mathbf{e}[n] \mathbf{e}^H[k] \mathbf{V}(\mathbf{z}, \mathbb{1}) \right] \\ &= \frac{1}{\sigma^2} \mathbf{X}^H[k] \mathbf{D}^H(\mathbf{z}, \mathbb{1}) \mathbf{V}(\mathbf{z}, \mathbb{1}) \end{aligned} \quad (66)$$

for $k = 0, \dots, N-1$.

The Fisher information matrix in (34) may be written as

$$\mathbf{J}_F(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{J}(\mathbf{z}, \mathbf{z}) & \boldsymbol{\gamma}^T \\ \boldsymbol{\gamma} & \mathbf{Q} \end{bmatrix}, \quad (67)$$

where $\boldsymbol{\gamma} = [\mathbf{J}(\mathbf{z}, \mathbf{x}[0]) \cdots \mathbf{J}(\mathbf{z}, \mathbf{x}[N-1])]$, and $\mathbf{Q} = \text{diag}(\mathbf{J}(\mathbf{x}[0], \mathbf{x}[0]), \dots, \mathbf{J}(\mathbf{x}[N-1], \mathbf{x}[N-1]))$. Therefore, the CRB matrix for the estimation of the mode parameters $\mathbf{z} = [z_1, \dots, z_p]$, which is the top left $p \times p$ block of $\mathbf{J}_F^{-1}(\boldsymbol{\theta})$ is

$$\begin{aligned} \text{CRB}(\mathbf{z}, \mathbb{I}) &= [\mathbf{J}(\mathbf{z}, \mathbf{z}) - \boldsymbol{\gamma}^T \mathbf{Q}^{-1} \boldsymbol{\gamma}]^{-1} \\ &= \left[\mathbf{J}(\mathbf{z}, \mathbf{z}) - \sum_{n=0}^{N-1} \mathbf{J}(\mathbf{z}, \mathbf{x}[n]) \mathbf{J}^{-1}(\mathbf{x}[n], \mathbf{x}[n]) \mathbf{J}(\mathbf{z}, \mathbf{x}[n]) \right]^{-1}. \end{aligned} \quad (68)$$

Using (64), (65) and (66) in (68), we have

$$\begin{aligned} \text{CRB}(\mathbf{z}, \mathbb{I}) &= \sigma^2 \left[\sum_{n=0}^{N-1} \mathbf{X}^H[n] \mathbf{D}^H(\mathbf{z}, \mathbb{I}) [\mathbf{I} - \mathbf{P}_{\mathbf{V}(\mathbf{z}, \mathbb{I})}] \mathbf{D}(\mathbf{z}, \mathbb{I}) \mathbf{X}[n] \right]^{-1}. \end{aligned} \quad (69)$$

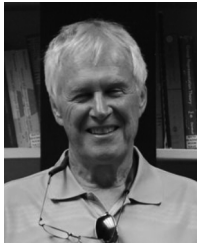
REFERENCES

- [1] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," in *Proc. IEEE Digital Signal Process. Workshop IEEE Signal Process. Edu. Workshop (DSP/SPE)*, Coolum, Queensland, Australia, Jan. 2011, pp. 289–294.
- [2] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.
- [3] P. P. Vaidyanathan and P. Pal, "Direct-MUSIC on sparse arrays," in *Proc. IEEE Int. Conf. Signal Process. Commun.*, Bangalore, India, Jul. 2012, pp. 1–5.
- [4] P. P. Vaidyanathan and P. Pal, "Why does direct-MUSIC on sparse-arrays work?," in *Proc. Rec. Asilomar Conf. Signals, Syst., Comput.*, Pacific Grove, CA, USA, Nov. 2013, pp. 2007–2011.
- [5] P. P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3592–3608, Aug. 2011.
- [6] R. Prony, "Essai expérimental et analytique sur les lois de la dilatabilité des fluides élastiques, et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alcool, à différentes températures," *J. L'École Polytechnique (Paris)*, vol. 1, no. 2, pp. 24–76, 1795.
- [7] R. Kumaresan, L. L. Scharf, and A. K. Shaw, "An algorithm for pole-zero modeling and spectral analysis," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 34, no. 3, pp. 637–640, Jun. 1986.
- [8] Y. Bresler and A. Macovski, "Exact maximum likelihood parameter estimation of superimposed exponential signals in noise," *IEEE Trans. Acoust. Speech, Signal Process.*, vol. 34, no. 5, pp. 1081–1089, Oct. 1986.
- [9] R. Kumaresan and D. W. Tufts, "Estimating the angles of arrival of multiple plane waves," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-19, no. 1, pp. 134–139, Jan. 1983.
- [10] R. Kumaresan, D. W. Tufts, and L. L. Scharf, "A Prony method for noisy data: Choosing the signal components and selecting the order in exponential signal models," *Proc. IEEE*, vol. 72, no. 2, pp. 230–233, Feb. 1984.
- [11] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, Mar. 1986.
- [12] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 7, pp. 984–995, Jul. 1989.
- [13] F. Belloni, A. Richter, and V. Koivunen, "Extension of root-MUSIC to non-ULA array configurations," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Toulouse, France, May 2006, vol. 4, pp. 897–900.
- [14] M. Rubsamen and A. B. Gershman, "Direction-of-arrival estimation for nonuniform sensor arrays: From manifold separation to Fourier domain MUSIC methods," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 588–599, Feb. 2009.
- [15] B. Friedlander and A. J. Weiss, "Direction finding using spatial smoothing with interpolated arrays," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 2, pp. 574–587, Apr. 1992.
- [16] B. Friedlander, "The root-MUSIC algorithm for direction finding with interpolated arrays," *Signal Process.*, vol. 30, no. 1, pp. 15–29, June 1993.
- [17] F. Gao and A. B. Gershman, "A generalized ESPRIT approach to direction-of-arrival estimation," *IEEE Signal Process. Lett.*, vol. 12, no. 3, pp. 254–257, Mar. 2005.
- [18] L. Coluccio, A. Eisenberg, and G. Fedele, "A Prony-like method for non-uniform sampling," *Signal Process.*, vol. 87, no. 10, pp. 2484–2490, Oct. 2007.
- [19] T. Peter, D. Potts, and M. Tasche, "Nonlinear approximation by sums of exponentials and translates," *SIAM J. Sci. Comput.*, vol. 33, no. 4, pp. 1920–1947, Aug. 2011.
- [20] A. Moffet, "Minimum-redundancy linear arrays," *IEEE Trans. Antennas Propag.*, vol. AP-16, no. 2, pp. 172–175, Mar. 1968.
- [21] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.
- [22] J. McClellan and D. Lee, "Exact equivalence of the Steiglitz-McBride iteration and IQML," *IEEE Trans. Signal Process.*, vol. 39, no. 2, pp. 509–512, Feb. 1991.
- [23] P. Pakrooh, L. L. Scharf, A. Pezeshki, and Y. Chi, "Analysis of Fisher information and the Cramér-Rao bound for nonlinear parameter estimation after compressed sensing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Vancouver, BC, Canada, May 2013, pp. 6630–6634.
- [24] P. Pakrooh, A. Pezeshki, L. L. Scharf, D. Cochran, and S. D. Howard, "Analysis of Fisher information and the Cramér-Rao bound for nonlinear parameter estimation after random compression," *IEEE Trans. Signal Process.*, vol. 63, no. 23, pp. 6423–6428, Dec. 2015.
- [25] P. Pakrooh, A. Pezeshki, and L. L. Scharf, "Threshold effects in parameter estimation from compressed data," in *Proc. IEEE Global Conf. Signal Inf. Process. (GlobalSIP)*, Austin, TX, USA, Dec. 2013, pp. 997–1000.
- [26] H. Achanta, S. Biswas, S. Dasgupta, M. Jacob, B. Dasgupta, and R. Mudumbai, "Coprime conditions for Fourier sampling for sparse recovery," in *Proc. IEEE 8th Sensor Array Multichannel Signal Process. Workshop (SAM)*, Jun. 2014, pp. 533–536.
- [27] L. L. Scharf, *Statistical Signal Processing*. Reading, MA, USA: Addison-Wesley, 1991.
- [28] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramér-Rao bound," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 37, no. 5, pp. 720–741, May 1989.
- [29] P. J. Schreier and L. L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals*. Cambridge, U.K.: Cambridge Univ. Press, 2010.
- [30] T. McWhorter and L. L. Scharf, "Cramer-Rao bounds for deterministic modal analysis," *IEEE Trans. Signal Process.*, vol. 41, no. 5, pp. 1847–1866, May 1993.



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