A Refinement-by-Superposition hp-Method for H(curl)- and H(div)-Conforming Discretizations

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Abstract—We present an application of refinement-by-superposition (RBS) *hp*-refinement in computational electromagnetics (CEM), which permits exponential rates of convergence. In contrast to dominant approaches to *hp*-refinement for continuous Galerkin methods, which rely on constrained-nodes, the multi-level strategy presented drastically reduces the implementation complexity. Through the RBS methodology, enforcement of continuity occurs by construction, enabling arbitrary levels of refinement with ease and without the practical (but not theoretical) limitations of constrained-node refinement. We outline the construction of the RBS *hp*-method for refinement with *H*(curl)- and *H*(div)-conforming finite cells. Numerical simulations for the 2-D finite element method (FEM) solution of the Maxwell eigenvalue problem demonstrate the effectiveness of RBS *hp*-refinement. An additional goal of this work, we aim to promote the use of mixed-order (low- and high-order) elements in practical CEM applications.

Index Terms—computational electromagnetics, continuous Galerkin, finite element method, higher order methods, *hp*-refinement, multi-level, refinement-by-superposition.

I. INTRODUCTION

GEOMETRIC discretization by quadrilaterals and hexahedra, while significantly more accurate with respect to degrees of freedom (DoFs) than modeling with triangles or tetrahedra [1], presents significant challenge to fully dynamic mesh adaptivity. In refinement with triangles and tetrahedra, local adaptivity directives propagate to a small set of neighboring elements, enabling the insertion of new DoFs without modification to the entire element structure. Similar refinement approaches with quadrilateral and hexahedral cells, however, would dictate global refinement, thereby destroying the utility of h-adaptivity. Inserting transition elements also poses significant challenge to the approximation quality, particularly for vectorial shape functions, which rapidly degrades as the unit vectors lose linear independence in the physical domain. Finally, application of a discontinuous Galerkin method, while evading this problem, among its other difficulties, generally requires more DoFs for the same level of accuracy.

As shown in [2]–[5], when the solution satisfies certain regularity conditions, p-refinement enables exponential convergence; when such conditions are not satisfied, however, the benefit of p-refinement is heavily degraded, reduced instead to algebraic convergence as in the case of pure h-refinement. Pure p-refinement, while effective in certain situations (e.g., [6]), is therefore insufficient in general, and as such, a combined approach with both h- and p-adaptivity is necessary to achieve

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exponential convergence for solutions with singularities or non-smooth behavior, motivating the need for more advanced and versatile approaches to h-refinement.

Previous works in CEM, for example, have demonstrated the potential of hp-adaptivity through hybrid meshes, e.g., in [7]. Most typically, however, to address the hp-adaptivity limitations the insertion of constrained-nodes, which—in contrast to true DoFs—are constrained to enforce continuity conditions with neighboring elements, is performed. Such approaches in CEM have shown significant performance increases and exponential convergence in the presence of singular solutions [8]–[13], but at the cost of high implementation complexity, impeding wide-scale adoption. Furthermore, such methods are usually limited in implementation to 1-irregular meshes (i.e., only one hanging node per edge), which, while not a severe limitation in practice, prevents arbitrary local refinement steps. Open-source libraries—such as deal.II [14]—have significantly simplified the implementation of hp-refinement codes, yet in some cases it might be inconvenient or undesirable to utilize third-party finite element method (FEM) libraries.

As such, we opt to extend the refinement-by-superposition (RBS) approach introduced in [15]–[17] for hierarchical basis functions, which demonstrated exponential convergence for scalar problems with C^0 finite elements, to H(curl)- and H(div)-conforming finite elements. Additional studies with C^0 finite elements and the RBS hp-method with adaptivity in [18] further motivate extensions of the method to CEM.

While the proposed approach significantly simplifies the implementation of hp-refinement infrastructure for applications in CEM, in contrast to more traditional refine-by-replacement (RBR) strategies, the proposed approach decreases the sparsity of the system matrices under both h- and p-refinements, whereas RBR only reduces sparsity under p-refinements. The proposed approach is therefore less suitable for application to very large problems with vast differences in scales. For applications of the approach to boundary element method (BEM) problems, e.g., surface integral equation (SIE) problems in CEM discretized by the method of moments (MoM), such considerations do not apply given the global nature of the Green's function, but would instead concern the increase in integration time due to the overlap of refinement layers.

The remainder of this paper is organized as follows. Section II details the construction of the RBS hp-method, covering the enforcement of the required continuity conditions (tangential or normal continuity) and ensuring linear independence after the insertion of descendant refinement layers. Section III examines application to an H(curl)-conforming discretization of the Maxwell eigenvalue problem. We examine a challenging eigenpair with a singular eigenfunction as studied in [19].

The presented approach yields exponential convergence of the eigenvalue with respect to the number of degrees of freedom (NDoFs), which, along with the ease of implementation, illustrates the practical value for applications in CEM.

II. Refinement-by-Superposition: Description and Construction

With an underlying hierarchical H(curl)- or H(div)-conforming basis, such as introduced in [20], exponential convergence may be achieved with suitable refinements by a collection of overlay meshes. This RBS approach yields the desired discretization by imposing homogeneous Dirichlet boundary conditions on the boundaries of the inserted descendant cells (i.e., the collection of overlay meshes) [15]–[17]. Continuity requirements, therefore, may be easily enforced for arbitrary levels of refinements (i.e., n-irregular meshes) and heterogeneity in the chosen orders of the hierarchical basis throughout the mesh. This enables an algorithmically straightforward and low-cost method to add hp-refinement capabilities for H(curl)- and H(div)-conforming discretizations by enforcing, respectively, tangential and normal continuity.

We approach the description explicitly from a 2-D perspective; however, the process generalizes trivially to 3-D. First, we classify each shape function in the following manner. In the case of an H(curl)-conforming discretization, we assign each shape function according to the properties of the non-zero tangential components at the boundary of the cell. These shape functions are classified into three categories: the node-functions, i.e., those functions with non-zero tangential components at only one node; the edge-functions, i.e., those functions with non-zero tangential components along one and only one edge; and the cell-functions, i.e., those functions which have no non-zero tangential components on the boundary. The existence of node-functions is only necessary when in 2-D, the axial component (i.e., perpendicular to the 2-D plane of the geometry) of the solution is non-zero, as in the examples in [6]. The same classification strategy, albeit according to the non-zero normal components at the boundary, is applied for seeking solutions in H(div). Naturally, the difficulty in inserting unknowns rests in the treatment of the edge-functions (and potentially the node-functions), while the cell-functions, which introduce no DoFs influencing the boundary, may be inserted or excised without non-local considerations.

Given some starting mesh, which, without loss of generality, we assume to be regular, refinement directives are then executed. Furthermore, while we focus on isotropic refinement, the superposition-based approach supports anisotropy in p-refinement directives trivially and, with modification, to anisotropic h-refinements [21]. For example, for a single h-refinement step applied to one cell in 2-D, four new child cells are inserted one refinement layer above the parent cell according to the constraint of isotropic refinement only. Note that geometrically, the child cells lie in the same physical space as the parent; rather, the designation of "above" is purely conceptual.

We have two simultaneous considerations in the process: continuity and linear independence. Continuity must be enforced due to potential non-uniformity in the polynomial degree of the basis on neighboring cells and the existence (or lack thereof) of child cells on the various refinement levels. Linear independence, on the other hand, is guaranteed by the proper delegation of DoFs between the refinement layers descended from the origin cell. DoFs must be deactivated on the parent cells and activated on the child cells, depending on the refinement levels of the cell and its neighbors.

The refinement and coarsening directives are applied for both cases through the assignment of the DoFs to the geometrical structures as mentioned above (the cells, the edges, the nodes, and, in 3-D, the faces).

A. Enforcing Continuity Requirements

The activation and deactivation of the DoFs on the boundaries of the cells follows a unified procedure based on [15]–[17]. For each layer in the discretization, from the origin layer to the highest refinement layer, DoFs are assigned to each of the geometrical elements according to the continuity requirements desired, in this case, tangential continuity for seeking a solution in H(curl) and normal continuity for H(div). Each cell, edge, node (when necessary in 2-D), and face (exclusively in 3-D) collects a list of DoFs, both active and inactive.

As the DoFs are accumulated, each one is matched as necessary with the associated shape functions on neighboring cells according to the vectorial direction and multi-index of the associated shape function. As opposed to the boundary-functions (i.e., node, face, and edge), the cell-functions automatically satisfy continuity requirements and therefore no special considerations are necessary except for those related to ensuring linear independence. The boundary-function DoFs are marked as active according to the existence of neighbors in the refinement level and the expansion order of those cells. Both restrictions are handled seamlessly and without distinction as the overall process amounts to traversing the geometrical entities in the discretization, which are assigned their maximal sets of associated DoFs, and activating only the DoFs according to the above compatibility conditions.

We summarize the procedure for activating DoFs based on the continuity requirements as follows:

1) For each cell, edge, node, and face in each refinement layer, collect the associated DoFs
   a) For H(curl), associate the DoFs based on the non-zero tangential components
   b) For H(div), associate the DoFs based on the non-zero normal components
2) Iterate through each refinement layer and each edge, node, and face
   a) If a suitable refinement neighbor exists, match the shape functions associated with the adjacent cells, activating only the fully matched DoFs and deactivating the rest

B. Eliminating Linear Dependence of the Hierarchical Refinements

For ensuring linear independence between overlapping shape functions, we prioritize the highest feasible refinement

level possible. For example, the cell-functions, which by definition satisfy the continuity requirements automatically, require deactivation on the parent cell and activation of the DoFs on the child cells. Unlike the handling of the edge- and node-functions, this transfer occurs without any queries to the discretization other than checking if the descendant cells exist.

Now, for the edge- and node-functions, additional care is necessary. In this case, the preference to delegate DoFs to the child cells is constrained by the refinement state of one or more neighbors of the cell. In other words, in 2-D, if a parent cell shares an edge with another refined parent cell, the DoFs on the parent edge may be transferred to the corresponding edges on the child refinement layer. Likewise, the deactivation of a node-function on the parent refinement layers requires that the corresponding node is surrounded by refined cells. In other words, as in [15]–[17], active geometrical components may not "overlap" with respect to the refinement layers.

We summarize the activation and deactivation of DoFs as follows:

1) On an h-refinement step, deactivate cell-functions on the parent cell and activate the cell-function DoFs on the child cell.
2) If a geometrical component (a node, edge, or face) on the descendant layer is active (i.e., it has associated active DoFs), deactivate the corresponding component on the parent layer

According to this procedure, a parent cell sufficiently surrounded by refined cells may be entirely deactivated to ensure linear independence and maximize the resolution of the approximation. In such cases, the sparsity of the resulting system is enhanced.

C. Summary of the Overall Approach

Illustrated in Fig. 1, the entire procedure, from enforcing the continuity requirements to ensuring linear independence, requires only the straightforward rules as summarized in the preceding subsections. Similarly to the descriptions of the RBS process in 1-D in [15]–[17], Fig. 1(a) summarizes the procedure for a 1-D domain, including the transfer of DoFs associated with lower refinement levels to the descendant layers and the ability to choose the expansion order p arbitrarily. Note that in 1-D, we have only the linear class of boundary-functions, i.e., at the boundary between two cells (across all the refinement levels), only one active boundary-DoF exists. The hierarchical basis functions illustrated in Fig. 1(a) and those used in the Numerical Results Section are based on the maximally-orthogonalized basis functions [20].

In 2-D (and 3-D), however, many active DoFs exist on the cell boundaries as a result of employing higher order boundary-functions. Depicted in Fig. 1(b), we demonstrate a similar refinement model as in the 1-D case. Unlike in the 1-D case, the depicted refinement in 2-D results in the enforcement of the domain boundary conditions propagating to the higher refinement levels when available. Furthermore, in this case, many of the parent cells retain a large number of DoFs assigned to the boundary due to the higher order boundary-functions. In Fig. 1(b), such occurrences are denoted
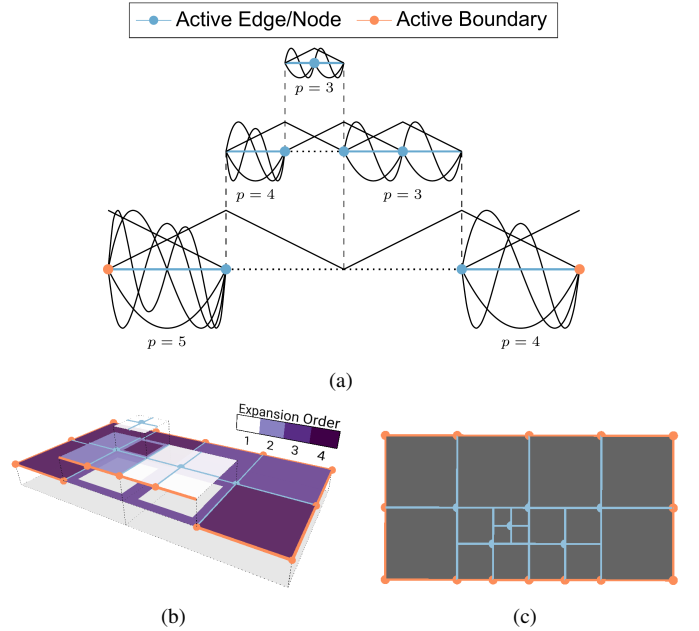

Fig. 1. The RBS hp-refinement activation and deactivation procedure. (a) The depiction of the process in 1-D. (b) The depiction of the process in 2-D. (c) An overhead perspective of the distribution of h-refinements applied in the 2-D example.

by the cells with solid boundaries and transparent interiors, in addition to the matching designations related to the active geometrical components as seen in Fig. 1(a).

Finally, for each cell located on a new refinement layer, an additional mapping is introduced, resulting in a succession of mapping operations. Note, however, that regardless of the curvature of the origin cell, all subsequent mappings from the reference cell to the child cell have constant Jacobian determinants, and may be handled with ease during integration.

III. NUMERICAL RESULTS

We now demonstrate the suitability of the RBS hp-refinement methodology by solving the following Maxwell eigenvalue problem (in variational form):

Find $U = \{\mathbf{u}_{hp}, \lambda_{hp}\} \in B_{hp} \times \mathbb{R}$ such that

$$a(\mathbf{u}_{hp}, \boldsymbol{\phi}_{hp}) = \lambda_{hp} m(\mathbf{u}_{hp}, \boldsymbol{\phi}_{hp}) \quad \forall \boldsymbol{\phi}_h \in B_{hp}, \quad (1)$$

for $B_{hp} \subset H(\text{curl}; \Omega)$, where $m(\mathbf{u}_{hp}, \boldsymbol{\phi}_{hp}) = \langle \mathbf{u}_{hp}, \boldsymbol{\phi}_{hp} \rangle$, and $a(\mathbf{u}_{hp}, \boldsymbol{\phi}_{hp}) = \langle \nabla_t \times \mathbf{u}_{hp}, \nabla_t \times \boldsymbol{\phi}_{hp} \rangle$. We further assert that the domain $\Omega \subset \mathbb{R}^2$ is terminated by the Dirichlet boundary condition $\mathbf{n} \times \mathbf{u}_{hp} = 0$ on $\partial\Omega$. Finally, $\mathbf{u}_{hp}$ is purely transversal.

While not exclusively applicable to eigenvalue problems with singularities, we study the approach for a 2-D cross-section of an L-shaped waveguide, shown in Fig. 2(a), which features many singular eigenfunctions, to demonstrate the capability to achieve exponential convergence in the presence of solution irregularity. We focus our analysis on the convergence of the smallest eigenvalue to an accurate numerical computation [19] of the benchmark problem originally proposed by [22]. The eigenfunction associated with this eigenvalue exhibits a singularity in the field at the reentrant corner, as seen in Fig. 2(b).
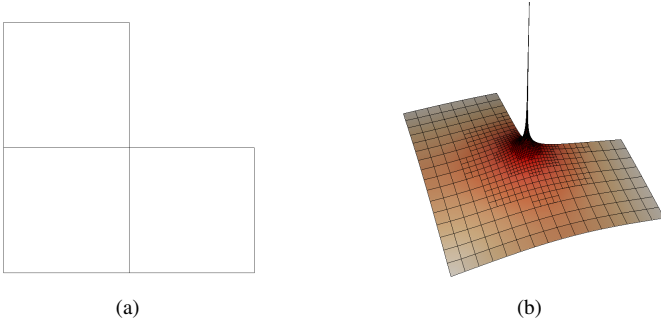
Fig. 2. The model and problem under study. (a) The initial discretization for the L-shaped domain. (b) The field magnitude of the first eigenfunction, illustrating the singularity at the reentrant corner.
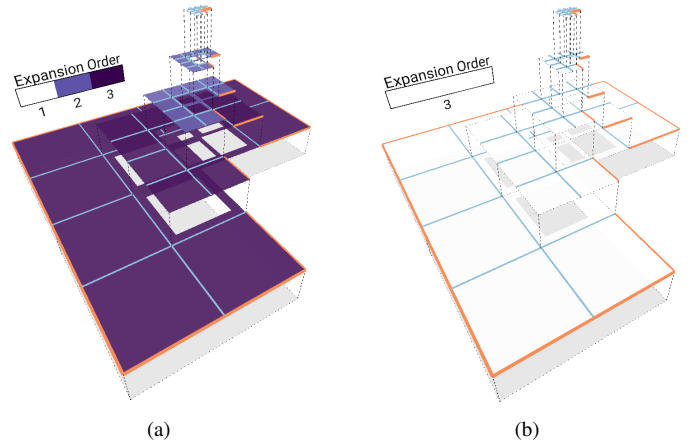

Fig. 3. Example discretizations for the RBS hp-method and the selectively h-refined comparison method with uniform p. (a) The RBS hp-method discretization with maximum and minimum expansion orders of three and one, respectively. (b) The RBS h-method with a uniform expansion order of three. The two discretizations have $L = 5$ refinement levels.
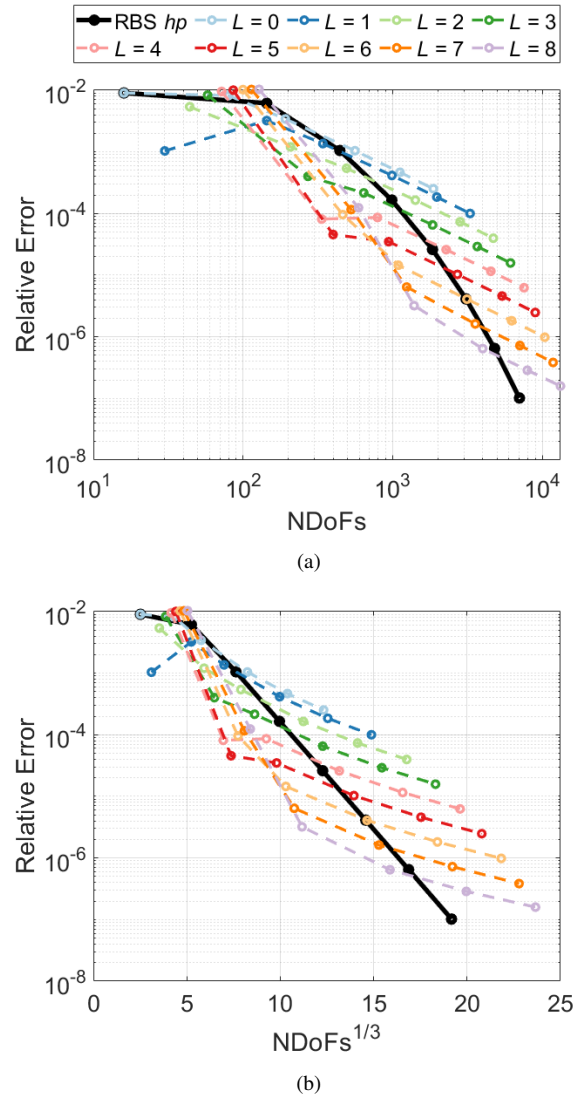
Following the procedure outlined in Section II, the initial discretization is successively refined about the reentrant corner. New refinement layers are inserted in groups with $p = 1$ and the expansion orders of each pre-existing cell are increased by one each iteration, resulting in an emphasis on h-refinements closer to the reentrant corner and an emphasis on p-refinements away from the reentrant corner. We note that this illustrative *a priori* refinement strategy is neither optimal nor adaptive. Adaptive strategies, such as in [19], may be applied in place of the illustrative refinement approach presented in this manuscript. A collection of discretizations with h-refinements targeting the reentrant corner (from $L = 0$ to $L = 8$ refinement levels) and global (i.e., uniform) increments in p serves as the comparison approach.

Example discretizations from each approach with five refinement layers are illustrated in Fig. 3. Fig. 3(a) depicts the progression from third-order field expansion to first-order while undergoing simultaneous h-refinements and Fig. 3(b) features the same level of h-refinement with homogeneous third-order field expansion.

Fig. 4 shows the convergence results for the first eigenvalue with the two approaches to refinement. RBS hp-refinement achieves exponential convergence while the successively p-refined discretizations at various levels of h-refinement ($L = 0$ to $L = 8$) provide only algebraic convergence. The linear trend with respect to NDoFs$^{1/3}$ as in Fig. 4(b) indicates the strong consistency of the exponential convergence.

IV. Conclusion

We have demonstrated the capability to achieve exponential convergence through an RBS hp-method in CEM. At the cost of reducing sparsity in FEM applications, the significant reduction in implementation complexity facilitates straightforward adoption of hp-refinement techniques with arbitrary levels of refinement.

When applied to the computation of the eigenvalue associated with a singular eigenfunction for $H(\mathrm{curl})$-conforming elements, the method delivers perfect exponential convergence while enforcing the tangential continuity requirements by construction rather than through constraint equations. Finally, the entire procedure directly applies to enforcement of normal continuity when $H(\mathrm{div})$-conforming elements are required and also extends to 3-D applications easily.

Fig. 4. Convergence of the first eigenvalue for the RBS hp-method and the comparison approach with h-refinement levels from $L = 0$ to $L = 8$ and increasing uniform expansion orders. (a) Double logarithmic representation. (b) log-cube-root representation.

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