B-Spline Entire-Domain Higher Order Finite Elements for 3-D Electromagnetic Modeling

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***Abstract*—A novel entire-domain finite element technique based on curvilinear hexahedral geometrical modeling using B-splines in conjunction with higher order hierarchical polynomial curl-conforming vector basis functions for field expansions is proposed for electromagnetic analysis of microwave devices. The effectiveness of the technique is evaluated in analysis of microwave cavities and it is compared with alternative reference solutions and with HFSS. Examples demonstrate large single-element (entire-domain) models with p-refined field distributions and better accuracy in curvature modeling than with the Lagrange interpolation. Extraordinary flexibility of the B-spline hexahedral elements, combined with higher order field expansions, allows accurate modeling of complex structures with sharp edges using a single finite element.**

***Index Terms*—Cavities, finite element methods (FEMs), microwave devices, spline functions.**

I. Introduction

THE finite element method (FEM) is a powerful numerical tool for electromagnetic (EM) modeling, especially when coupled with the higher order computational approach [1]. Recently, curvilinear elements for geometrical modeling of EM structures have been extensively investigated [2]–[8]. Polynomial parametrizations of the curvilinear elements (e.g., Lagrange polynomials, Bézier curves, and splines) [2]–[7] are mostly used, but rational polynomial functions (e.g., rational Bézier curves and non-uniform rational B-splines or NURBS) have also been adopted [7], [8], although they are more complex and less stable [7].

Although some works do combine higher order curvilinear geometrical modeling with higher order basis functions, none of them seem to exploit the full potential of the two methodologies with both orders adopted arbitrarily (but reasonably) high and independently from each other—except when Lagrange elements are used, and none offer assessment of benefits and penalties resulting from the parametrizations more complex than Lagrange interpolation.

This letter proposes a novel FEM technique using B-spline hexahedral elements in conjunction with higher order hierarchical polynomial curl-conforming vector basis functions, and evaluates the improvements over the existing methods—in analysis of microwave cavities. This is the first demonstration of (a couple of wavelengths) large curved B-spline elements with p-refined field distributions of high (e.g., seventh) approximation orders in EM. We also show for the first time that the ridged cavity (challenging due to complex shape and reentrant corners) can be efficiently modeled using a single finite element (an entire-domain model). Although automatic applicability of the proposed method is somewhat hindered, mainly by the absence of required hexahedral meshers, the method can be efficiently used whenever the problem can be reduced to swept geometry, or in conjunction with the semi-automatic "top-down" meshing approach [9]. The proposed modeling can also motivate development of new curvilinear hexahedral meshing algorithms.

II. Theory and Implementation

We use the following recurrent formula to define the B-spline functions because it facilitates simple implementation:

$$\begin{aligned} B_{i,1}(u) &= 1, u_i \le u \le u_{i+1} \quad \text{and} \\ B_{i,1}(u) &= 0, \quad \text{elsewhere}, \\ B_{i,m}(u) &= \frac{u-u_i}{u_{i+m-1}-u_i} B_{i,m-1}(u) \\ &\quad + \frac{u_{i+m}-u}{u_{i+m}-u_{i+1}} B_{i+1,m-1}(u), m > 1 \end{aligned} \tag{1}$$

where $0 \le i \le n, n > 0$, and $U = (u_0, u_1, \ldots, u_{n+m})$ is a non-decreasing sequence of real numbers. If division by zero occurs, the term is replaced by zero. The function $B_{i,m}(u)$ is called the i-th B-spline of order m and degree $m-1$ with respect to the knot vector U. The length of U determines the number of functions of the given order, whereas the multiplicities of the knots (knots with the same value) determine the smoothness of the spline functions. Division by zero occurs when U contains multiplicities. The following equations hold for a standard clamped uniform knot vector:

$$\begin{aligned} u_i &= 0, 0 \le i \le m-1, \\ u_i &= i-m+1, m \le i \le n, \quad \text{and} \\ u_i &= n-m+2, n+1 \le i \le n+m. \end{aligned} \tag{2}$$

Using B-splines, we define a smooth parametric hexahedron introducing a mapping $\mathbf{r} : (u,v,w) \rightarrow (x,y,z), (u,v,w) \in [-1,1] \times [-1,1] \times [-1,1]$ (cubical parent domain), such that

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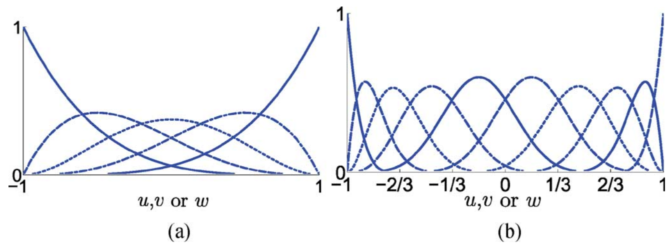

Fig. 1. B-spline functions of order $m = 5$ used for entire-domain modeling of a spherical cavity: (a) Model II with the flat knot vector $(-1, 1)$ and (b) Model III with the knot vector $(-1, -2/3, -1/3, 0, 1/3, 2/3, 1)$.

it is interpolatory at the specified points of the global Cartesian space. To simplify the parameterization (without loss of generality) we employ the same order of B-splines $(m_u = m_v = m_w = m)$ and the same knot vectors in all directions. A point within a hexahedron is thus defined by

$$\mathbf{r}(u,v,w) = \sum_{i,j,k=0}^{n} B_{i,m}(u) B_{j,m}(v) B_{k,m}(w) \mathbf{C}_{i,j,k} \quad (3)$$

where $B_{i,m}, B_{j,m}, B_{k,m}$ are the splines over the same knot vector and $\mathbf{C}_{i,j,k}$ are the position vectors of the control points, found by solving the following system of equations:

$$\mathbf{r}_l = \sum_{i,j,k=0}^{n} B_{i,m}(u^l) B_{j,m}(v^l) B_{k,m}(w^l) \mathbf{C}_{i,j,k}, \quad l = 1, \ldots, K \quad (4)$$

where $K = (n+1)^3$, and with $\mathbf{r}_l$ and (u^l, v^l, w^l) being the (global) position-vectors of the interpolation points of the solid and their (local) parametric coordinates, respectively. The choice of interpolation points and a knot vector depends on the particular solid that needs to be parametrized.

III. Numerical Results and Discussion

As the first example, consider a spherical air-filled metallic cavity of radius $R_0 = 1$ cm. We parametrize the sphere using a direct analytical mapping given by

$$(x,y,z) = R_0 \Big(u\sqrt{1 - v^2/2 - w^2/2 + v^2w^2/3}\,, \; v\sqrt{1 - u^2/2 - w^2/2 + u^2w^2/3}, \; w\sqrt{1 - u^2/2 - v^2/2 + u^2v^2/3} \Big) \quad (5)$$

and the appropriate B-spline functions of order $m = 5$ (degree $m - 1 = 4$), as shown in Fig. 1, with the knot vector for which the number of interpolation points matches the number of functions. Five models are used for comparisons: tetrahedral model [10], HFSS model (pre-meshed with 3 496 tetrahedra with 2nd order basis functions to achieve percentage error close to 1% for the first mode), Model I (with a 4th order Lagrange hexahedron [3]), and Models II and III [with novel B-spline elements; Model II is a 5th order B-spline model with 125 interpolation points $(n = 4)$, and Model III is a 5th order B-spline model with 1000 interpolation points $(n = 9)$].

TABLE I
Relative Error in Calculating k_0 for a Spherical Cavity: Comparison of p-Refined Entire-Domain B-Spline Models II [Fig. 1(a)] and III [Fig. 1(b)] With the Tetrahedral Model [10], HFSS, and Lagrange Model I [3]

	|Error| [%]					
Mode ↓	Tetrah. [10]	HFSS	Lagr. Model I	B-spline Model II/Model III		
Unkn. →	300	3,496 t	450	240	450	756
Rel. time	N/A	2.16	1	0.19/0.41	0.67/1.42	1.90/4.02
TM_{010}	2.04	1.15	0.25	0.0605 / 0.0121	0.0363 / 0.0105	0.0363 / 0.0106
TM_{111}^{even}	2.11	1.15	0.25	0.0605 / 0.0121	0.0363 / 0.0105	0.0363 / 0.0106
TM_{111}^{odd}	2.44	1.15	0.25	0.0605 / 0.0121	0.0363 / 0.0105	0.0363 / 0.0106
TM_{021}	2.02	1.00	0.02	0.12 / 0.0979	0.0792 / 0.0071	0.0242 / 0.0071
TM_{121}^{even}	2.99	1.00	0.02	0.12 / 0.0979	0.0792 / 0.0071	0.0242 / 0.0071
TM_{121}^{odd}	3.20	1.00	0.02	0.12 / 0.0979	0.0792 / 0.0581	0.0242 / 0.0075
TM_{221}^{even}	4.34	1.00	0.23	0.3628 / 0.2502	0.0977 / 0.0581	0.0977 / 0.0075
TM_{221}^{odd}	4.59	1.00	0.23	0.3628 / 0.2502	0.0977 / 0.0581	0.0977 / 0.0075
TE_{011}	1.33	0.39	0.28	0.3151 / 0.2623	0.2218 / 0.1691	0.0582 / 0.0127
TE_{111}^{even}	0.47	0.39	0.28	0.3151 / 0.2623	0.2218 / 0.1691	0.0582 / 0.0127
TE_{111}^{odd}	1.25	0.39	0.28	0.3151 / 0.2623	0.2218 / 0.1691	0.0582 / 0.0127

Table I gives the percentage errors (with respect to the analytical solution) of the computed resonant-mode free space wavenumbers, k_0. The number of unknowns for the Models II and III is increased by p-refinement $(N_u = N_v = N_w = N)$ [3] from $N = 5$ to $N = 7$. Note that all three B-spline models are actually entire-domain models. We observe that (a) the proposed B-spline elements have excellent convergence properties, (b) the error obtained with Model II (for which the error in numerically computing the volume of the element is 0.1%) is lower than that with Model I, especially when looking at the first three resonant modes, (c) when the average relative error of the first 11 modes is compared for Models I and II (with 450 unknowns), Model II yields on average 1.75 times lower error, (d) Model III (which computes the volume with a 0.0001% error) achieves very low error even with only 240 unknowns, and (e) the error obtained with Model III is on average over 1.6 times (with 450 unknowns) and over 5 times (with 756 unknowns) lower than the error achieved with Model II. Note that the low (oscillating) errors (e.g., Model I for modes 4 to 6) fall in the error margin for the particular mode and the particular numerical discretization, and they cannot be used as a general estimate of the solution accuracy [3]. Note also that B-spline models yield up to two orders of magnitude lower errors than the tetrahedral model [10]. Also shown in the table are computational times expressed in relative units (Rel. time), where the relative time amounting to 1 stands for the time required to solve Model I. It can be seen that B-spline models are the most efficient, yielding up to 80% decrease in the computational time (Model II with 240 unknowns). Finally, as it turned out to be rather difficult for us to obtain high accuracy using HFSS, the shown respective relative time is for the single adaptive pass in HFSS.

As the second example, consider a ridged air-filled metallic cavity, shown in Fig. 2(a). The cavity belongs to a class of swept geometries, i.e., its shape can be extruded from the base shown in the w-cut in Fig. 2(b), and all of its coordinate lines except one (v) are straight. The whole cavity can thus be easily modeled by

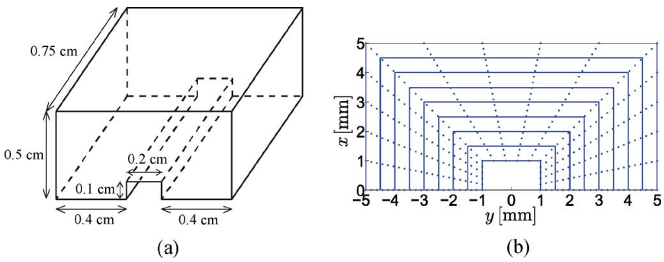

Fig. 2. Entire-domain modeling of a rigded cavity: (a) geometry and (b) $u-v$ coordinate lines in one w-cut of the B-spline parametrization of the cavity.

TABLE II
COMPUTED k_0 FOR THE RIDGED CAVITY IN FIG. 2(A): COMPARISON OF THE SINGLE-ELEMENT (ENTIRE-DOMAIN) B-SPLINE MODEL [FIG. 2(B)] AND FIVE ALTERNATIVE SOLUTIONS

		Computed k_0 [cm^{-1}]								
Mode →	Unkn. ↓	1	2	3	4	5	6	7	8	9
T_1 [10]	267	4.941	7.284	7.691	7.855	8.016	8.593	8.906	9.163	9.679
T_2 [10]	671	4.999	7.354	7.832	7.942	7.959	8.650	8.916	9.103	9.757
L_1 [3]	81	5.088	7.471	7.903	7.967	8.019	9.001	9.111	9.169	10.08
L_2	1,730	5.092	7.477	7.855	7.894	8.019	8.902	9.097	9.113	10.02
HFSS	3,017 t	5.091	7.469	7.853	7.878	8.019	8.863	8.900	9.087	10.00
B-spl.	276	5.093	7.163	7.886	8.278	8.324	9.101	9.290	9.706	10.39

one element. To parametrize the cavity, the same spline families [of order $m=2$ and $U=(-1,-1/3,1/3,1)$] are used in all directions. This results in $K=4^3$ interpolation points, which are equally spaced along the u- and w- directions, and define a ridge along the v- direction. The geometry in this case is modeled exactly and the main source of error arises from numerical (polynomial) approximations of the fields, which are particularly tricky near the sharp reentrant corners of the ridge.

The numerical results are summarized in Table II, where the B-spline results are compared with two tetrahedral models (T_1 and T_2) from [10], two Lagrange (trilinear) models [5-element model L_1 and 6-element (h-refined 3-element) model L_2] from [3], and with HFSS (with 3 017 pre-meshed tetrahedra). We consider the (h-refined) L_2 and HFSS results to be of the highest accuracy, as it is known that h-refinement is superior to p-refinement in the presence of singular fields. We can conclude from the table that B-spline entire-domain results are excellent (accurate to three significant digits) for the dominant mode (which is usually the most important one) and very accurate for the higher modes. Note that the B-spline results are obtained in 60% less amount of time when compared to the HFSS solution (single pass and meshing time excluded). B-spline results are also better or comparable with T_1 and T_2 results using similar or lower number of unknowns. While the L_1 model uses fewer unknowns, it also uses 5 elements (h-refinement), which adds to the complexity of the FEM formulation in this example. To the best of our knowledge, this is the first entire-domain model of a structure geometrically as complex and numerically as challenging as the ridged cavity and the first demonstration that such a structure can be efficiently and accurately modeled using a single finite element with a simple B-spline representation in combination with the higher order FEM.

IV. CONCLUSION

This letter has proposed and examined a novel entire domain numerical technique for efficient modeling of electromagnetic structures based on B-spline geometrical representation and the higher order FEM. The effectiveness of the proposed technique has been tested on examples of FEM simulations of a spherical cavity and a ridged cavity (eigenvalue problems). It has been shown that the method enables modeling of curved geometrical shapes with excellent precision, as shown in the spherical cavity example, resulting in by up to two orders of magnitude lower errors in computed dominant resonant wavenumbers when compared to lower order methods and by one order of magnitude lower errors when compared to an alternative higher order method (using Lagrange elements). It has also been demonstrated (for the first time) that, due to flexibility of the B-spline geometrical modeling, a ridged cavity can be modeled using a single finite element while achieving a very good accuracy in computation of the first several resonant wavenumbers when paired with the efficient higher order FEM method. In addition, although entire-domain modeling demonstrated in this work can be considered as an extreme higher order computation, since the real-world engineering tasks will certainly require complex meshes, the entire-domain B-spline FEM modeling may open new ways of thinking regarding the possible element shapes and sizes in general. It may also aid a future solution to the problem of automatic higher order curved hexahedral mesh generation.

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