Higher Order FEM-MoM Domain Decomposition for 3-D Electromagnetic Analysis

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***Abstract*—A novel higher order domain decomposition (DD) method based on a hybridization of the finite element method (FEM) and method of moments (MoM) is proposed for three-dimensional (3-D) modeling of antennas and scatterers. The method implements multiple FEM domains within a global unbounded MoM environment, based on the surface equivalence theorem. The presented analyses of 3-D and two-dimensional (2-D) finite periodic arrays of inhomogeneous dielectric scatterers demonstrate excellent accuracy, convergence, and efficiency of the new FEM-MoM-DD technique, and a substantial reduction in the memory requirements and computational time when compared to the higher order MoM solution.**

***Index Terms*—Domain decomposition (DD), electromagnetic (EM) analysis, finite element method (FEM), higher order modeling, hybrid methods, method of moments (MoM), numerical techniques.**

I. Introduction

THE continuing and growing demand for computational tools that can handle full-wave simulations of larger and more complex electromagnetic (EM) problems has recently led to rapid development of novel domain decomposition (DD) algorithms. These algorithms allow splitting of the original, large problem into a number of smaller ones, which can be analyzed independently and then stitched together by some sort of local or integral boundary conditions, yet yielding in the process a rigorous solution of Maxwell's equations for the problem. This way, the computational burden can be tremendously reduced. Additionally, DD algorithms inherently allow parallelization since each of the smaller problems can be handled by a separate machine (or processor core).

Among a variety of possible and existing DD algorithms, we concentrate on those based on the three-dimensional (3-D) finite element method (FEM) in the frequency domain. Such algorithms constitute the mainstream of recent research efforts in this area. Advanced FEM-based DD techniques generally rely on the finite element tearing and interconnecting (FETI) [1] and alike algorithms [2]–[6], which use the Robin-type transmission condition to enforce continuity of the fields on the interdomain interfaces and allow efficient iterative substructuring, thus avoiding potential numerical problems that can arise in more conventional DD approaches. Some recent important improvements of DD algorithms include the introduction of nonconformal grids, which allow for a different triangulation on either side of interdomain interfaces [2]–[4], and the dual primal formulation, where some directly imposed continuity constraints across interdomain interfaces indirectly enforce all other constraints by using dual variables (Lagrange multipliers) [5], [6].

The FEM indeed very efficiently deals with problems consisting of inhomogeneous (and complex), arbitrarily shaped objects. When coupled with a boundary integral (BI) technique, which introduces an exact termination to numerically truncate and close the FEM computational domain, it yields powerful and versatile FE-BI hybrids, e.g. [7]. Some algorithms explicitly combine FEM volumetric modeling with solutions of surface integral equations (SIEs) based on the method of moments (MoM), e.g., [8], and such hybrids are hence also referred to as FEM-MoM techniques. Only most recently, however, more attention has been drawn to hybrid domain decomposition algorithms where different methods are employed to render solutions in distinct domains. Some examples of progress in this direction include a domain decomposition method based on the boundary element method (BEM) and FEM reported in [9] and an extremely robust two-level domain decomposition method using a FEM-based domain decomposition within a global hybrid domain decomposition method presented in [10]. However, all existing, hybrid, and other FEM-based DD tools appear to be low-order (small-domain) techniques. In general, it is well established that the higher order (large-domain) computational approach, which utilizes higher order field/current basis functions defined on large curvilinear geometrical elements, e.g., [11], can substantially enhance the accuracy and efficiency of EM modeling.

This letter proposes a novel higher order FEM-MoM domain decomposition method for 3-D EM modeling of antennas and scatterers. It capitalizes on our previous work in the hybrid FEM-MoM methodology [8], where a hybridization of FEM [12] and MoM [13] techniques is reported, both in the framework of the higher order large-domain modeling. In our FEM-MoM hybrid, the MoM part provides much greater modeling versatility and potential for applications than just as a BI closure to the FEM part. In addition, the way our hybridization is realized theoretically allows incorporation of multiple

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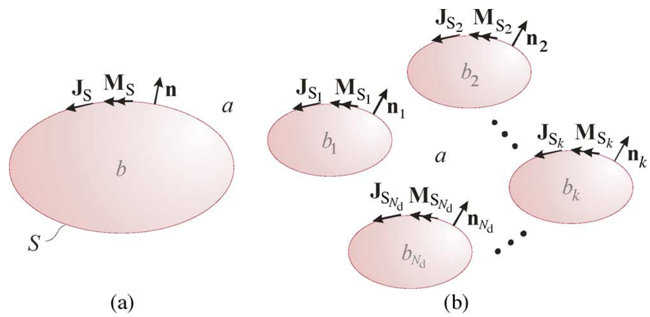

Fig. 1. Domain decomposition based on the surface equivalence theorem: (a) single subdomain and (b) extension to multiple subdomains.

FEM subdomains in a global, generally unbounded, MoM domain. The principal thrust of the present letter is numerical implementation and evaluation of this multiple-FEM-domains-in-MoM-environment hybrid concept, which is based on the surface equivalence principle and in which we analyze the subdomains using the FEM, independently, by applying unit current densities to equivalent sources, and then solve for the actual current sources by the MoM, discretizing a set of SIEs. The result is a higher order FEM-MoM-DD technique. To the best of our knowledge, this is the first higher order FEM-based domain decomposition method; when compared to the existing DD techniques, it provides the benefits of the higher order modeling and, in many cases, a feasible alternative to the recent solutions based on the Robin transmission condition. When compared to the higher order MoM-SIE and FEM-MoM (or FE-BI) techniques, on the other hand, it brings about the common DD advantages.

Section II of the letter presents the theoretical background and numerical implementation of the new higher order FEM-MoM-DD technique. As the details of the FEM-MoM hybridization are given in [8], we focus here on theoretical and implementation aspects of the surface equivalence theorem and inclusion of multiple FEM domains in a global MoM environment, as the core components of our DD method. In Section III, the technique is validated and its accuracy, convergence, and efficiency evaluated and discussed in characteristic examples of 3-D and two-dimensional (2-D) finite periodic arrays of inhomogeneous dielectric scatterers.

II. Theory and Implementation

A. Domain Decomposition Based on the Surface Equivalence Theorem

Consider a subdomain in a time-harmonic EM system situated in a generally unbounded homogeneous linear medium. The subdomain is of arbitrary shape, contains arbitrary linear inhomogeneities, and is bounded by a closed surface S (subdomain boundary), as shown in Fig. 1(a). In what follows, we refer to the exterior and interior of S as regions a and b, respectively, and denote by $\mathbf{n}$ the outward looking unit normal on it.

Based on the surface equivalence theorem, we place the equivalent surface electric current, of density $\mathbf{J}_{\mathrm{S}} = \mathbf{n}\times\mathbf{H}^a$, and equivalent surface magnetic current, of density $\mathbf{M}_{\mathrm{S}} = -\mathbf{n}\times\mathbf{E}^a$ (Huygens' sources) such that they support the original total fields in region a, $\mathbf{E}^a$ and $\mathbf{H}^a$, when all field excitations in region b are switched off, and annihilate the fields in region b. The fields in region a are thus driven solely by the (known) excitations in region a and (unknown) equivalent surface currents on the boundary S. Additionally, changing the signs of the equivalent surface currents and switching off the excitations in region a preserves the original fields in region b, $\mathbf{E}^b$ and $\mathbf{H}^b$, and annuls the fields in region a. This way, the fields in region b are driven solely by the (known) excitations in region b and the same (unknown) equivalent surface currents on the boundary S. Most importantly, the surface equivalence theorem enables an independent EM analysis of subdomain b and the rest of the system. Essentially, a linear relation between the equivalent surface currents and fields in region b (often referred to as a "numerical Green's function") can be found independently from the rest of the problem.

A straightforward extension of the situation in Fig. 1(a) to include multiple subdomains and boundary surfaces depicted in Fig. 1(b) represents the theoretical basis for a domain decomposition technique, where each of the subdomains b_k, $k = 1, 2, \ldots, N_{\mathrm{d}}$, can be analyzed independently, by a suitable method of choice, providing the relation between the fields in the subdomain and unit equivalent surface currents on its boundary, S_k. The currents are found solving the problem in domain a, which basically becomes a well-defined scattering problem. In our hybrid DD technique, we simply choose the FEM for analysis of subdomains (regions b) and MoM-SIE for analysis of the rest of the system (region a).

B. Inclusion of Multiple FEM Domains in a Hybrid FEM-MoM Technique

The surface equivalence theorem is already embraced as the theoretical foundation for establishing a set of coupled electric/magnetic field integral equations (EFIE/MFIE), required for analysis of homogeneous dielectric domains in our higher order MoM-SIE technique [13]. Hence, the main idea behind the construction of the DD method proposed in this letter is to exploit the existing MoM environment to solve for the unknown equivalent currents, $\mathbf{J}_{\mathrm{S}k}$ and $\mathbf{M}_{\mathrm{S}k}$ in Fig. 1(b), while replacing the homogeneous dielectric domains with arbitrarily complex and inhomogeneous regions in which the numerical Green's function is found by the higher order FEM. This latter step is carried out in a DD fashion technically equivalent to calculating the inner products $\langle \mathbf{j}_{\mathrm{S}_i}, \mathbf{E}^b(\mathbf{j}_{\mathrm{S}_j})\rangle$ and $\langle \mathbf{j}_{\mathrm{S}_i} \times \mathbf{j}_{\mathrm{S}_j}, \mathbf{n})\rangle$ over the FEM-MoM interfaces [8] independently for each of the subdomains and possibly simultaneously for all of them, where $\mathbf{j}_{\mathrm{S}i}$ and $\mathbf{j}_{\mathrm{S}j}$ are the higher order hierarchical polynomial vector testing and basis functions, respectively, defined on generalized curved quadrilateral MoM elements. The former step (solution for surface currents), on the other side, requires appropriate alterations to the MoM-SIE PMCHW formulation [11].

In our FEM-MoM-DD numerical procedure, the subdomain b_k in Fig. 1(b) is meshed using (large) generalized curved hexahedral FEM elements with higher order hierarchical polynomial vector bases so that $b_k = b_{k1} \bigcup b_{k2} \ldots \bigcup b_{kN_k}$. Outer faces of the hexahedra, S_{kj}, $j = 1, 2, \ldots, N_{Sk}$, belonging to the subdomain boundary, $S_k = S_{k1} \bigcup S_{k2} \ldots \bigcup S_{kN_{Sk}}$, are defined as (large) MoM quadrilaterals (carrying the equivalent surface currents $\mathbf{J}_{\mathrm{S}kj}$ and $\mathbf{M}_{\mathrm{S}kj}$). Within the MoM table of dielectric domains, each subdomain b_k is registered as a unique "special" dielectric domain that is algorithmically distinguished from the "regular" dielectric domains and can only appear on one of the sides of the MoM quadrilaterals. The electric and magnetic tangential-field boundary conditions for each of the MoM quadrilaterals whose one side is declared to belong to the "special" domain, i.e., quadrilaterals belonging to S_k, are formally modified from their original PMCHW form to equations with currents $\mathbf{J}_{\mathrm{S}k}$ and $\mathbf{M}_{\mathrm{S}k}$ over S_k and field $\mathbf{E}^b$ throughout subdomain b_k (one of the multiple FEM domains) as unknowns [8]. This technically reduces to setting the combined EFIE/MFIE equations in our MoM code on only one side of such quadrilaterals and calculating the MoM impedances (using Galerkin testing procedure) with no further adjustments. In the final step, the impedances linked with the testing and basis functions (when both belong to boundary MoM quadrilaterals associated with the same "special" dielectric domain) are augmented with corresponding previously calculated inner products. The described alterations pertain to all dielectric domains declared as "special" and associated MoM quadrilaterals on their boundaries. By solving the MoM matrix system, we solve for the unknown surface currents on the boundaries of all domains, and subsequently we can calculate the required electromagnetic quantities in all domains.

III. Results and Discussion

As the first example, consider a 3-D finite array of lossless coated cubical dielectric scatterers shown in the inset of Fig. 2. Although the main purpose of this example is validation and evaluation of the presented DD technique, note that 3-D (and 2-D) periodic arrays of scatterers also have great potential for applications in the context of metamaterials engineering, where their structure and arrangement influence the metamaterial performance [14]. Fig. 2 presents the monostatic radar cross section (RCS) of the array. The new higher order FEM-MoM-DD technique is employed with a geometrical model consisting of one FEM cubical hexahedron (of side length a and dielectric constant $\varepsilon_{\mathrm{r}1}$) and 6 FEM "cushions" in the form of pyramidal frusta (of dielectric constant $\varepsilon_{\mathrm{r}2}$) representing the coating (of thickness t), onto which six MoM square patches are attached (the mesh can be seen in Fig. 2). The FEM part of the structure, consisting of seven elements, is modeled (and computed) only once, and the MoM part is periodically translated seven times to build a symmetric spatial arrangement of eight cubes, equidistantly spaced at distances d, forming the 3-D array in Fig. 2.

The DD results in Fig. 2 are given for two FEM-MoM discretizations illustrating a p-refinement of the solution. In the first discretization, FEM field expansion orders are 4 (in all directions) in the inner cubical element, 1 in the cushions that model the thin coating in direction perpendicular to the cube faces, and 5 in the other two directions, while current expansion orders for MoM quadrilaterals are 4. These orders are 5, 2, 6, and 5, respectively, in the p-refined model. The resulting numbers of elements (written as FEM-MoM), unknowns (FEM-MoM), and total computational times (using an IBM ThinkPad T60p notebook computer with Intel T7200 CPU running at 2.0 GHz) are

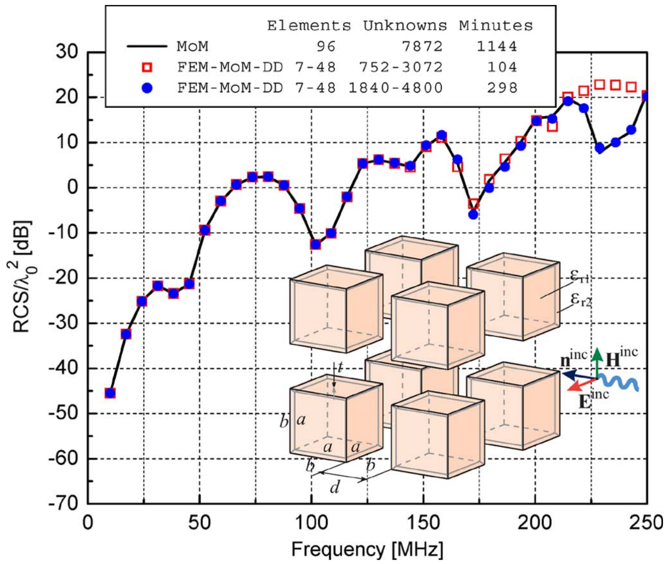

Fig. 2. Normalized monostatic RCS of a 3-D array of coated dielectric cubes (λ_0 is the free-space wavelength): convergence and efficiency analysis of the higher order FEM-MoM-DD with p-refinement and comparison of results with the reference MoM solution. ($a = 1$ m, $b = 1.04$ m, $t = 2$ cm, $d = 56$ cm, $\varepsilon_{\mathrm{r}1} = 4$, $\varepsilon_{\mathrm{r}2} = 2.5$).

given in the figure legend. We observe that the higher order hybrid FEM-MoM-DD solution accurately matches the reference MoM-SIE solution [13] and that it quickly converges when the structure is p-refined and the number of unknowns increased. Noting that the FEM calculation is carried out independently from MoM in the new method, we conclude that only the highest number of unknowns, i.e., the MoM one, in the solutions dictates the required memory resources, which are roughly proportional to the squares of the number of unknowns (for storage of the full MoM matrix). On the other hand, both computations commensurately contribute to the overall computational time. We thus conclude that the new method (the p-refined model) requires approximately 63% less memory and 74% less computational time than the reference MoM solution in this example, while maintaining the same accuracy.

As the second example, aimed at demonstrating the effectiveness of the new technique in treating curved and continuously inhomogeneous scatterers, Fig. 3 gives bistatic RCS results for a 2-D array of dielectric spheres (shown in the inset), whose dielectric constant changes in the radial direction linearly from 1 at the sphere surfaces to 6 at their centers. Each sphere is modeled by seven curvilinear hexahedral FEM elements of the second geometrical order, that is, by one small sphere-like hexahedron, $a/20$ in radius, at the center and six "cushion"-like hexahedra between the central sphere and the scatterer surface, onto which six curvilinear quadrilateral MoM patches are attached. FEM field expansion orders are 4 in the elements constituting the sphere and 3 on the quadrilateral patches. In a p-refined model, these orders are 5 and 4, respectively, and a very quick convergence of the results is observed in Fig. 3 (note that the same solution is also obtained with respective approximation orders increased to 6 and 5). Of course, the reference higher order MoM-SIE technique [13] is not directly applicable to this structure. Note that this example is illustrative of the necessity

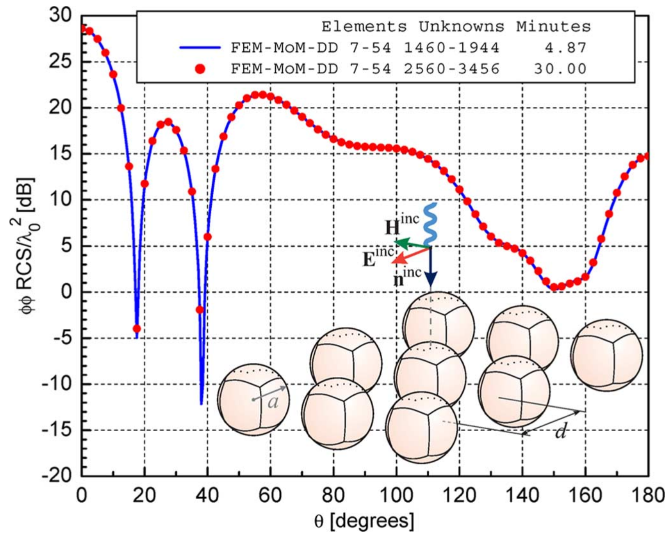

Fig. 3. Higher order FEM-MoM-DD simulations of the normalized bistatic RCS of a 2-D 3×3 array of continuously inhomogeneous lossless dielectric spheres (sphere radii are $a = 10$ cm, surface-to-surface distances between spheres are $d = 6$ cm), whose dielectric constant changes in a linear fashion from 1 at the sphere surfaces to 6 at their centers, at 1235 MHz.

and advantage of the higher order representation to accurately model the field behavior due to both the curvature of the large elements and their inhomogeneity.

IV. Conclusion

This letter has proposed a novel higher order FEM-MoM domain decomposition method for 3-D EM analysis. The method is based on numerical implementation of multiple FEM domains within a global unbounded MoM environment, which, in turn, is theoretically founded on the surface equivalence theorem. It represents the first higher order FEM-based DD method. The validity, accuracy, convergence, and efficiency of the new technique have been demonstrated in examples of 3-D and 2-D finite periodic arrays of inhomogeneous dielectric scatterers, in which very effective large-domain meshes of scatterers with small numbers of large FEM and MoM elements and p-refined field and current approximations are utilized, and only their MoM parts simply multiplied to achieve periodicity. The DD method enables a substantial reduction in the memory requirements and computational time when compared to higher order MoM solutions (when available).

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