

**An Introduction to
Numerical Modeling of the Atmosphere**

David A. Randall

Contents

| | |
|---|-----------|
| 1 Introduction | 3 |
| 1.1 What is a model? | 3 |
| 1.2 Elementary models | 4 |
| 1.3 Numerical models | 4 |
| 1.4 Physical and mathematical errors | 5 |
| 1.5 Discretization | 6 |
| 1.6 Physically based design of mathematical methods | 7 |
| 1.7 The utility of numerical models | 9 |
| 1.8 Where we are going in this book | 10 |
| 2 The basic equations in vector form | 12 |
| 2.1 Introduction | 12 |
| 2.2 The equation of motion | 12 |
| 2.2.1 Converting to a rotating frame of reference | 12 |
| 2.2.2 Forces | 15 |
| 2.2.3 Apparent gravity | 15 |
| 2.3 The continuity equation | 17 |
| 2.4 The thermodynamic energy equation | 17 |
| 2.5 The mechanical energy equation | 19 |
| 2.6 Total energy conservation | 20 |
| 2.7 The vertically integrated pressure-gradient force | 21 |
| 2.8 Problems | 22 |
| 3 Finite-difference approximations to derivatives | 23 |
| 3.1 Finite-difference quotients | 23 |
| 3.2 A fourth-order-accurate approximation | 27 |
| 3.3 A systematic way to construct finite-difference approximations to derivatives | 28 |
| 3.3.1 A family of schemes | 28 |
| 3.3.2 A generalization for use with nonuniform grids | 32 |
| 3.3.3 A further generalization to higher-order derivatives | 36 |
| 3.3.4 Extension to two dimensions | 37 |
| 3.4 Summary | 42 |

| | | |
|----------|--|-----------|
| 3.5 | Problems | 42 |
| 4 | Some time-differencing schemes | 44 |
| 4.1 | Introduction | 44 |
| 4.2 | A family of schemes | 45 |
| 4.3 | Discretization error | 46 |
| 4.4 | Explicit schemes | 49 |
| 4.5 | Implicit schemes | 52 |
| 4.6 | Iterative schemes | 54 |
| 4.7 | What's next? | 55 |
| 4.8 | Problems | 57 |
| 5 | The oscillation and decay equations | 58 |
| 5.1 | What a difference an i makes | 58 |
| 5.2 | Computational stability | 59 |
| 5.3 | The oscillation equation | 60 |
| 5.3.1 | The solution of the continuous oscillation equation | 60 |
| 5.3.2 | Amplitude errors and phase errors | 61 |
| 5.3.3 | Non-iterative two-level schemes for the oscillation equation | 63 |
| 5.3.4 | Iterative schemes for the oscillation equation | 65 |
| 5.3.5 | The leapfrog scheme for the oscillation equation | 67 |
| 5.3.6 | How the computational initial condition influences the initial amplitude of the computational mode | 69 |
| 5.3.7 | Ad hoc damping of computational modes in time | 70 |
| 5.3.8 | The stability of the leapfrog scheme for the oscillation equation | 71 |
| 5.3.9 | The second-order Adams-Bashforth Scheme for the oscillation equation | 75 |
| 5.3.10 | A survey of time differencing schemes for the oscillation equation | 77 |
| 5.4 | The decay equation | 77 |
| 5.5 | Damped oscillations | 82 |
| 5.6 | Nonlinear damping | 83 |
| 5.7 | Summary | 87 |
| 5.8 | Problems | 88 |
| 6 | Riding along with the air | 91 |
| 6.1 | The Lagrangian form | 91 |
| 6.2 | The advective form | 92 |
| 6.3 | The continuity equation | 94 |
| 6.4 | The flux form | 94 |
| 6.5 | Characteristics | 95 |
| 6.6 | Discussion | 98 |
| 7 | The upstream scheme | 99 |

| | | |
|-----------|---|------------|
| 7.1 | From here to there | 99 |
| 7.2 | The discretization error of the upstream scheme | 100 |
| 7.3 | The domain of dependence | 101 |
| 7.4 | Interpolation and extrapolation | 103 |
| 7.5 | Checking the computational stability of the upstream scheme | 105 |
| 7.5.1 | The direct method | 105 |
| 7.5.2 | The energy method | 106 |
| 7.5.3 | von Neumann's method | 107 |
| 7.6 | Including multiple wave numbers | 112 |
| 7.7 | How periodic boundary conditions come into play | 114 |
| 7.8 | Does the solution improve if we refine the grid? | 116 |
| 7.9 | Summary | 118 |
| 7.10 | Problems | 119 |
| 8 | “Forward in time” advection schemes | 120 |
| 8.1 | Accuracy and stability of a family of advection schemes | 120 |
| 8.2 | Matsuno time-differencing with centered space differencing | 123 |
| 8.3 | The Lax-Wendroff scheme | 123 |
| 8.4 | The Takacs scheme | 126 |
| 8.5 | Implicit schemes for the advection equation | 127 |
| 8.6 | Problems | 127 |
| 9 | Generalizing to two horizontal dimensions | 128 |
| 9.1 | Introduction | 128 |
| 9.2 | More about the Laplacian | 128 |
| 9.2.1 | Approximations to the Laplacian on rectangular grids | 128 |
| 9.2.2 | Integral properties of the Laplacian | 130 |
| 9.3 | Two-dimensional advection | 131 |
| 9.3.1 | Why be square? | 134 |
| 9.4 | Summary | 136 |
| 9.5 | Problems | 136 |
| 10 | Finite-volume methods | 139 |
| 10.1 | Definitions of vector operators | 139 |
| 10.2 | How is discrete conservation defined? | 140 |
| 11 | Conservative advection schemes | 142 |
| 11.1 | Continuous advection in one dimension | 142 |
| 11.2 | Conserving mass | 143 |
| 11.3 | Conserving an intensive scalar | 144 |
| 11.4 | An advective form | 145 |
| 11.5 | Conserving a function of an advected scalar | 146 |
| 11.6 | Lots of ways to interpolate | 148 |

| | | |
|-----------|--|------------|
| 11.7 | Fixers | 150 |
| 11.8 | A flux form of the upstream scheme | 150 |
| 11.9 | Problems | 152 |
| 12 | Computational dispersion | 156 |
| 12.1 | Centered space differencing and computational dispersion | 156 |
| 12.2 | More about computational dispersion | 159 |
| 12.3 | The effects of fourth-order space differencing on the phase speed | 167 |
| 12.4 | Space-uncentered schemes | 167 |
| 12.5 | Even- and odd-order schemes | 169 |
| 12.6 | Sign-preserving and monotone schemes | 171 |
| 12.7 | Hole filling | 173 |
| 12.8 | Flux-corrected transport | 175 |
| 12.9 | A survey of some advection schemes that you might run into out there | 178 |
| 12.10 | Summary | 178 |
| 13 | Lagrangian and semi-Lagrangian advection schemes | 180 |
| 13.1 | Lagrangian schemes | 180 |
| 13.1.1 | Smoothed particle hydrodynamics | 181 |
| 13.1.2 | Slippery sacks | 182 |
| 13.2 | Semi-Lagrangian schemes | 183 |
| 13.2.1 | The basic idea | 183 |
| 13.2.2 | More accurate semi-Lagrangian schemes | 185 |
| 13.2.3 | Remapping schemes | 187 |
| 14 | Just relax | 188 |
| 14.1 | Introduction | 188 |
| 14.2 | A continuous one-dimensional boundary-value problem | 189 |
| 14.3 | Fourier methods to solve the Poisson equation | 190 |
| 14.4 | Finite-difference methods to solve the Poisson equation | 190 |
| 14.5 | Jacobi relaxation | 193 |
| 14.6 | Gauss-Seidel relaxation | 197 |
| 14.7 | The alternating-direction implicit method | 200 |
| 14.8 | Multigrid methods | 200 |
| 14.9 | Summary | 204 |
| 14.10 | Problems | 205 |
| 15 | It's only dissipation. (But I like it!) | 208 |
| 15.1 | Introduction | 208 |
| 15.2 | A simple explicit scheme | 210 |
| 15.3 | An implicit scheme | 212 |
| 15.4 | The DuFort-Frankel scheme | 214 |
| 15.5 | Hyperdiffusion | 215 |

| | |
|--|------------|
| 15.6 Summary | 216 |
| 15.7 Problems | 216 |
| 16 The shallow-water equations | 217 |
| 16.1 Introduction | 217 |
| 16.2 Energy conservation with the shallow water equations | 219 |
| 16.3 Vorticity and potential vorticity | 220 |
| 16.4 Finite difference schemes for the one-dimensional case, with no rotation | 220 |
| 16.5 The nondivergent barotropic vorticity equation | 223 |
| 17 Making waves | 225 |
| 17.1 The shallow-water equations | 225 |
| 17.2 Staggered grids for the shallow water equations | 227 |
| 17.3 Wave propagation on two-dimensional staggered grids | 230 |
| 17.4 Dependence on the radius of deformation | 236 |
| 17.5 Other meshes | 239 |
| 17.6 Time-differencing schemes for the shallow-water equations | 241 |
| 17.6.1 Centered in space and time | 241 |
| 17.6.2 Implicit schemes | 244 |
| 17.6.3 The forward-backward scheme | 246 |
| 17.7 Summary and conclusions | 248 |
| 17.8 Problems | 248 |
| 18 Walls, real and imaginary | 251 |
| 18.1 Introduction | 251 |
| 18.2 Real walls | 251 |
| 18.3 Advection at inflow boundaries | 253 |
| 18.4 Outflow boundaries | 261 |
| 18.5 Energy fluxes at outflow boundaries | 266 |
| 18.6 Advection on inhomogeneous grids | 270 |
| 18.6.1 What does the downstream signal look like? | 272 |
| 18.6.2 How much of the incoming signal is reflected, and how much is transmitted? | 275 |
| 18.6.3 Choosing the weights at the seam | 276 |
| 18.7 What goes where? | 277 |
| 18.8 The effects of a mean flow | 281 |
| 18.9 Summary | 282 |
| 18.10 Problems | 282 |
| 19 Conservative schemes for the one-dimensional nonlinear shallow-water equations | 284 |
| 19.1 Properties of the continuous equations | 284 |
| 19.2 The spatially discrete case | 287 |

| | |
|--|------------|
| 19.3 Summary | 295 |
| 19.4 Problems | 295 |
| 20 Equation sets | 296 |
| 20.1 Sound waves | 296 |
| 20.2 Noise-cancelling equation sets | 297 |
| 21 Stairways to heaven | 298 |
| 21.1 The third dimension is special | 298 |
| 21.2 Introduction to vertical coordinate systems | 299 |
| 21.3 The basic equations in height coordinates | 299 |
| 21.4 Transformation to generalized vertical coordinates | 302 |
| 21.4.1 Transforming the horizontal pressure-gradient force | 302 |
| 21.4.2 Transforming the vertical pressure-gradient force | 304 |
| 21.4.3 The transformed equations of horizontal and vertical motion | 306 |
| 21.4.4 Transforming the continuity equation | 307 |
| 21.5 Energy conservation with generalized vertical coordinates | 310 |
| 22 Vertical coordinates for quasi-static models | 313 |
| 22.1 The equation of horizontal motion | 313 |
| 22.2 The hydrostatic equation | 313 |
| 22.3 The vertically integrated HPGF | 314 |
| 22.4 The HPGF in the potential vorticity equation | 317 |
| 22.5 Vertical mass flux for a family of vertical coordinates | 318 |
| 22.6 A survey of vertical coordinate systems | 320 |
| 22.6.1 Height | 321 |
| 22.6.2 Pressure and log pressure | 326 |
| 22.6.3 Terrain-following coordinates | 330 |
| 22.6.4 The eta-coordinate | 335 |
| 22.6.5 Potential temperature and entropy | 337 |
| 22.6.6 Hybrid sigma-theta coordinates | 341 |
| 22.6.7 Summary of vertical coordinate systems | 344 |
| 22.7 Problems | 344 |
| 23 Vertical differencing | 346 |
| 23.1 Vertical staggering | 346 |
| 23.2 Conservation of total energy with continuous pressure coordinates | 348 |
| 23.3 Conservation of total energy with continuous sigma coordinates | 350 |
| 23.4 Total energy conservation as seen in generalized coordinates | 354 |
| 23.5 Conservation properties of vertically discrete models using sigma-coordinates | 358 |
| 23.5.1 The horizontal pressure-gradient force | 360 |
| 23.5.2 The thermodynamic energy equation | 361 |
| 23.5.3 The mechanical energy equation | 363 |

| | |
|---|------------|
| 23.5.4 Total energy conservation | 365 |
| 23.5.5 The problem with the L grid | 366 |
| 23.6 Summary and conclusions | 369 |
| 23.7 Problems | 369 |
| 24 When the advector is the advectee | 370 |
| 24.1 Introduction | 370 |
| 24.2 Scale interactions and nonlinearity | 370 |
| 24.2.1 Aliasing error | 371 |
| 24.2.2 Almost famous | 371 |
| 24.2.3 A mathematical view of aliasing | 372 |
| 24.3 Advection by a variable, non-divergent current | 375 |
| 24.4 Aliasing instability | 378 |
| 24.4.1 An example of aliasing instability | 379 |
| 24.4.2 Analysis in terms of discretization error | 383 |
| 24.4.3 Discussion | 385 |
| 24.5 Fjortoft's Theorem | 386 |
| 24.6 Kinetic energy and enstrophy conservation in two-dimensional non-divergent flow | 392 |
| 24.7 The effects of time differencing on conservation of squares | 406 |
| 24.8 Conservative schemes for the two-dimensional shallow water equations with rotation | 408 |
| 24.9 Angular momentum conservation | 411 |
| 24.10 Summary | 412 |
| 24.11 Problems | 412 |
| 25 Finite differences on the sphere | 415 |
| 25.1 Introduction | 415 |
| 25.2 Spherical coordinates | 415 |
| 25.2.1 Vector calculus in spherical coordinates | 415 |
| 25.2.2 The "pole problem" | 417 |
| 25.2.3 Polar filters | 422 |
| 25.3 The Kurihara grid | 424 |
| 25.4 Displaced poles | 425 |
| 25.5 Grids Based on Map Projections | 425 |
| 25.6 Composite grids | 429 |
| 25.7 Unstructured spherical grids | 430 |
| 25.8 Summary | 435 |
| 25.9 Problems | 435 |
| 26 Spectral methods | 436 |
| 26.1 Introduction | 436 |
| 26.2 Transform pairs | 436 |

| | | |
|-----------|--|------------|
| 26.3 | Differentiation | 437 |
| 26.4 | Truncation | 437 |
| 26.5 | Spectral differentiation in terms of finite-differences | 440 |
| 26.6 | Solving linear equations with the spectral method | 441 |
| 26.7 | Solving nonlinear equations with the spectral method | 443 |
| 26.8 | The transform method | 444 |
| 26.9 | Spectral methods on the sphere | 446 |
| 26.9.1 | Spherical harmonics | 446 |
| 26.9.2 | Truncation | 448 |
| 26.10 | Spherical harmonic transforms | 450 |
| 26.11 | How it works | 451 |
| 26.12 | Semi-implicit time differencing | 452 |
| 26.13 | Conservation properties and computational stability | 453 |
| 26.14 | The accuracy of spectral models | 453 |
| 26.15 | Physical parameterizations | 455 |
| 26.16 | Moisture advection | 455 |
| 26.17 | Linear grids | 456 |
| 26.18 | Reduced linear grids | 456 |
| 26.19 | Summary | 456 |
| 26.20 | Problems | 457 |
| 27 | Finite-Element Methods | 459 |
| 27.1 | Problems | 460 |
| 28 | Concluding discussion | 461 |
| | Appendices | 462 |
| A | A Demonstration that the Fourth-Order Runge-Kutta Scheme Really Does Have Fourth-Order Accuracy | 462 |
| B | Vectors, Coordinates, and Coordinate Transformations | 471 |
| B.1 | Physical laws and coordinate systems | 471 |
| B.2 | Scalars, vectors, and tensors | 471 |
| B.3 | Differential operators | 474 |
| B.4 | Vector identities | 476 |
| B.5 | Spherical coordinates | 478 |
| B.5.1 | Vector operators in spherical coordinates | 478 |
| B.5.2 | Horizontal and vertical vectors in spherical coordinates | 479 |
| B.5.3 | Derivation of the gradient operator in spherical coordinates | 481 |
| B.5.4 | Applying vector operators to the unit vectors in spherical coordinates | 482 |
| B.6 | Solid body rotation | 483 |
| B.7 | Formulas that are useful for two-dimensional flow | 484 |

| | | |
|--------------|--|------------|
| B.8 | Vertical coordinate transformations | 485 |
| B.8.1 | Basics | 485 |
| B.8.2 | Some useful operators | 486 |
| B.9 | Concluding summary | 487 |
| C | Spherical Harmonics | 488 |
| | Bibliography | 499 |